A Different Viewpoint
by Gerald Fitton

The Earth Rising
as seen from the Moon

Using different points of view is a writing skill.
Marks are awarded for the use of viewpoint.

The National Curriculum

There are two senses in which the phrase, “Change one’s View” can be used. One is the sort of change of view experienced by St Paul on the Road to Damascus where he changed from Hating to Loving.

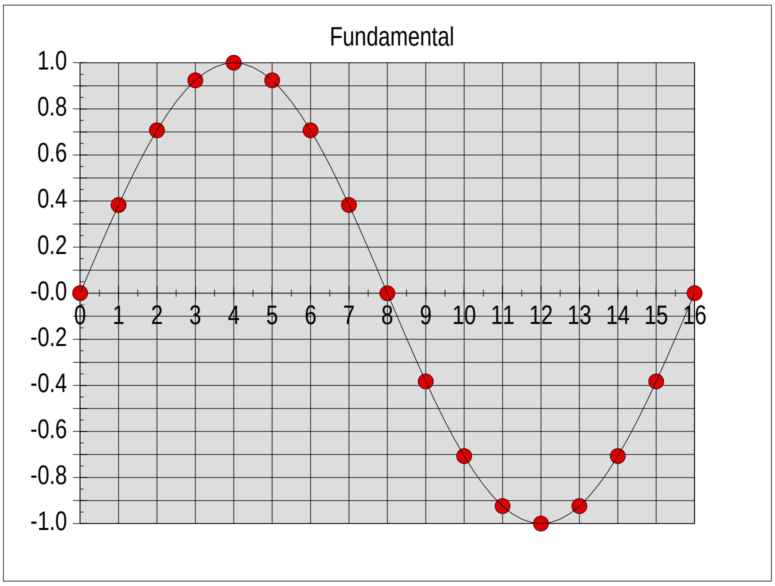
My article this month is not about the sort of change of opinion which moves a slider along the same scale (e.g. a scale running from Hate to Love or a scale which has graduations representing Worst, Worse, Bad, Good, Better and Best) but it is better described by the phrase, “A Change of Viewpoint”. Perhaps a couple of examples of what I mean will help.

I remember the first time I saw a photograph of the Earth taken from outer space. Until then the only pictures I had seen of the Earth were artists’ impressions. Without exception, all these artists’ impressions depicted the Earth without any clouds. Similarly I was greatly surprised when I first saw satellite photographs of the area where I have lived for 50 years.

In this article I shall discuss (with worked PipeDream spreadsheets) such a radical change of Viewpoint. In my example I shall start with the Viewpoint that Time is my independent variable; then I shall change my Viewpoint by introducing a different independent variable.

A Sound Wave

The graph, ‘Fundamental’, has, as its independent variable, Time; the dependent variable is the instantaneous air Pressure at a point in space. This graph shows one cycle of of a sound wave for which the Time axis is marked 0 to 16 instead of from 0 to 1. Why? Be patient. All will be explained.



Although the Time units could be anything at all, let's think of them as milliseconds. A complete cycle of this sound wave takes place over a periodic time of 16 milliseconds. Although this graph shows only one cycle of this sound wave, please imagine that it carries on repeating this same 16 millisecond cycle well into the future - as it did in the past.

The formula I have used for this 'Fundamental' sine wave is $p = \sin(2\pi t/16) = \sin(1\pi t/8)$.

The PipeDream spreadsheet I have used to draw it is shown in the second screenshot.

::201.$.2012.GC1201.Files.PipeDream.sin01a

sin(1*pi*A3/8)

	A	B	C	D
1				
2	n	sin(1*pi*n/8)	n	sin(1*pi*n/8)
3	0.00	0.0000	0	0.0000
4	0.25	0.0980	1	0.3827
5	0.50	0.1951	2	0.7071
6	0.75	0.2903	3	0.9239
7	1.00	0.3827	4	1.0000
8	1.25	0.4714	5	0.9239
9	1.50	0.5556	6	0.7071
10	1.75	0.6344	7	0.3827
11	2.00	0.7071	8	-0.0000
12	2.25	0.7730	9	-0.3827
13	2.50	0.8315	10	-0.7071
14	2.75	0.8819	11	-0.9239
15	3.00	0.9239	12	-1.0000
16	3.25	0.9569	13	-0.9239
17	3.50	0.9808	14	-0.7071
18	3.75	0.9952	15	-0.3827
19	4.00	1.0000	16	0.0000

The smooth curve is drawn using the first two columns (which have a 0.25 milliseconds time interval) whereas the red points are the points obtained by plotting columns C and D.

The Physicist or Sound Engineer may look at this periodic waveform of Pressure against Time and deduce many interesting things from it. However, we mere human beings do not experience this phenomenon as pressure variations. Our perception of a sound wave is not one which is comparable in any way with the way we sense Time dependent pressure variations in our hands and arms when using a hammer drill. The way in which we perceive the Pressure variations of a sound wave having a 16 millisecond period is as a pure sinusoidal tone with a frequency of 62.5 Hz.

We have changed our Viewpoint from one of Time dependence (the air Pressure is Time dependent) to one in which we regard the Frequency of the sound wave as all important.

GCSEs and A Levels

When we plot graphs at GCSE level they are all about Variables. We might plot a graph, y against x for $y = ax^2 + bx + c$ having been given numerical values for a, b and c. This graph is a parabola. The Variables are x and y; a, b and c are called Parameters.

Also at GCSE level we might plot a graph of $y = A\sin(\omega t + \psi)$ for different values of A, ω and ψ. This graph is a sine wave similar to the one in the graph called 'Fundamental'.

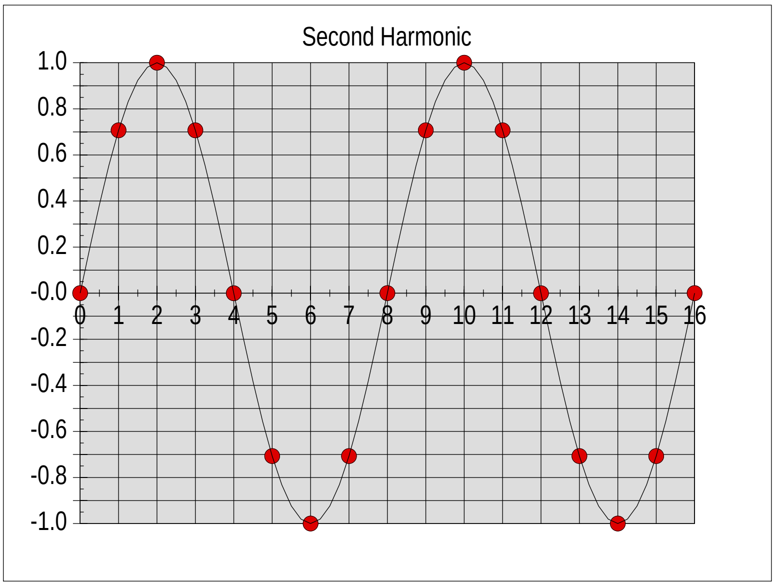
When we plot graphs at A level we no longer concentrate on variables such as x, y or t. We change our Viewpoint; we concentrate on the effect of changing the Parameters, a, b, c for the parabola and A, ω, ψ for the sine wave.

Sine Wave Parameters

The three Parameters of a sine wave, A, ω and ψ, change the shape of the graph. Let's consider how these changes in the Parameters affect the shape of our Fundamental graph.

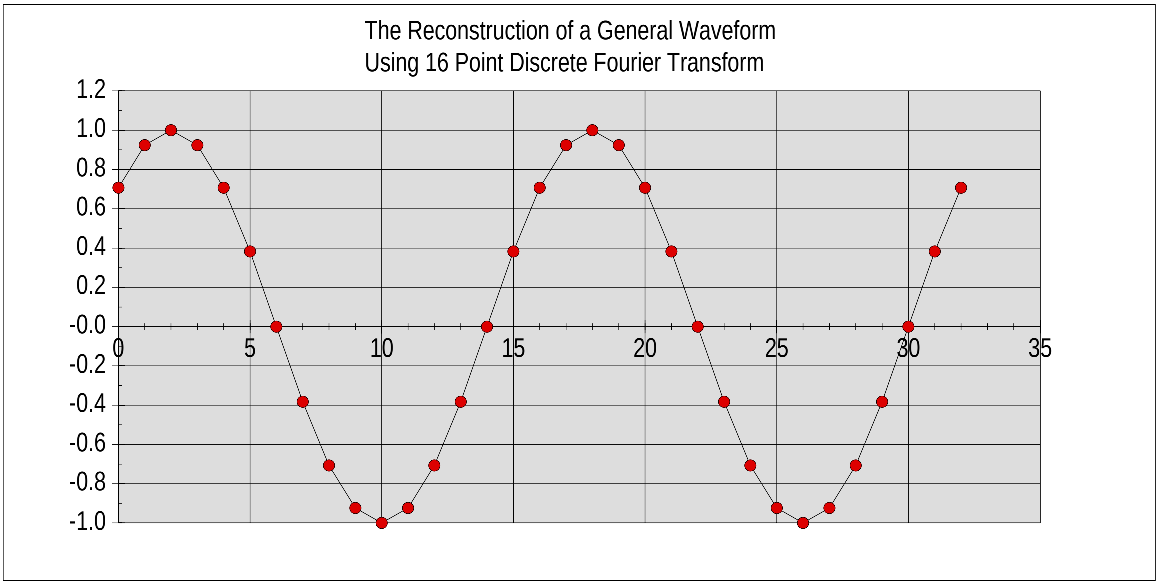
The effect of changing the Parameter A is relatively easy to describe. As A increases from 1.0 the maximum value increases. We experience this subjectively as an increase in the volume of the musical tone. We call A the Amplitude of the sine wave.

The Parameter ω is called the Frequency. The pitch of the tone we hear rises as ω is increased and falls as ω is reduced. Doubling the value of ω doubles the Frequency of the note we hear from, say, 62.5 Hz to 125 Hz.



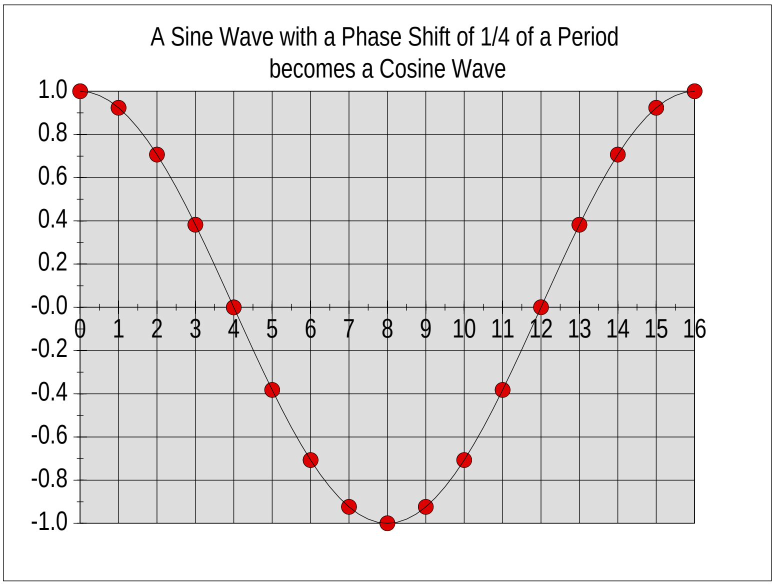
Have a look at the graph labelled Second Harmonic. You will see that over 16 milliseconds the air Pressure wave has completed two cycles rather than the single cycle of the graph labelled Fundamental. The formula for this Second Harmonic graph is $p = \sin(2\pi t/8)$ rather than the $p = \sin(1\pi t/8)$ of the Fundamental. We have doubled the value of ω.

The parameter ψ is, perhaps, the hardest to explain.



In the graph called 'The Reconstruction of a General Waveform', I have included two cycles of a sine wave which, as before, has Amplitude 1.0 and a 16 millisecond period. However, when the time (along the horizontal axis) is $t = 0$, the Pressure (the vertical axis of the graph) is not zero as it was before but it is about $p = 0.7$. This shift of the graph to the left ($p = 0$ when $t = -2$) is called the Phase Shift (or Phase Offset) of the sine wave. A single ear can do nothing with such a Phase Shift but, if we receive two copies of the sine wave having different Phase Shifts, one sound wave in each ear, then we can use the perceived Phase Difference to determine the apparent direction from which the sound originates. Our aural mechanism, which includes the use of our brain, is very sensitive to the Phase Difference which we receive between our two ears. Isn't our brain clever?

If we Phase Shift our sine wave through 1/4 of a 16 ms period, our sine wave turns into the corresponding cosine wave! The formula for this wave is $p = \sin(\pi(t+4)/8) = \cos(\pi t/8)$.



Finding the Fourier Transform

Our ears (or rather the whole aural system including the way our brains processes sound waves) receive a Time dependent Pressure waveform and then, using a very clever algorithm, we, our ears and brains, convert this Time dependent Viewpoint to one in which we experience only the Frequency or Frequencies of the sounds we hear.

There is a Mathematical algorithm for Changing our Viewpoint from Time dependence to the Viewpoint for which Frequency rather than Time is the Independent Variable. From a Mathematical Viewpoint we are trying to extract 'm', the Frequency, from the formula $p = \sin(\pi mt/8)$. The name given to this extraction of the Frequency from the Pressure Waveform is called 'Finding the Fourier Transform'.

I shall try to explain, with a somewhat simplified example, the way this algorithm is used to extract a set of frequencies, the values of 'm'(in our case the Natural Numbers, 0 to 8), from a general periodic Pressure Waveform $p = f(t)$ for which p is a function of the Time, t.

Rather than consider a general waveform at the moment, I shall concentrate on the waveform given by $p = \sin(4\pi t/16) = \sin(2\pi t/8)$. This is the Waveform I have called the Second Harmonic. The graph of this function appears earlier in this article. The Frequency which we wish to extract is $m = 2$, the 2 cycles per 16 milliseconds which appears in the formula $p = \sin(2\pi t/8)$.

The Fourier Transform method of extraction for our 16 point waveform is as follows:

Create a set of sine waveforms having the format $q = \sin(\pi nt/8)$ using values of n from $n = 0$ to $n = 8$. The 9 Waveforms, q, are 9 different functions because of the 9 different values of 'n' whereas (in this example - a more complex example later) the Waveform p is only a single function having one single value of 'm'. In our example the Waveform from we wish to extract the Frequency, m, is $p = \sin(2\pi t/8)$; the value of $m = 2$.

Using each of the 9 waveforms for q in turn, multiply $q \times p$ and integrate the function $q \times p$ over a single 16 millisecond cycle of the Pressure Waveform. Nearly all of these integrals will return zero but when the particular q is chosen for which $n = 2$, namely $q = \sin(2\pi t/8)$, the integral of $q \times p$ returns not zero but, wonder of wonders, it returns the number 8.

We have found the Fourier Transform of $p = \sin(2\pi t/8)$. This Fourier Transform has the simple result that the Frequency, $m = 2$ cycles per 16 milliseconds.

Box Out - The Maths Bit

The Maths which extracts the Frequencies from the Time dependent waveform requires some basic familiarity with the integration of trigonometrical functions. In our case we need to integrate the information in the Pressure Waveform over a complete cycle in order to find the Frequency.

For those of you who want to get into this 'Maths Bit', the extraction depends upon the fact that the integral of $\sin(\pi nt/8) \times \sin(\pi mt/8)$ over a whole period (exactly a whole period) is nearly always zero. It becomes something other than zero only when $n = m$.

It is important to note that ‘m’ and ‘n’ are not Real Numbers (from the set ℝ) but that they have to be Natural Numbers (‘counting numbers’ from the set ℕ). Only when ‘m’ and ‘n’ are Natural Numbers such as 0, 1, 2, 3, 4, 5, 6, 7, 8 does the function, p = sin(πmt/8) have the same period of 16 milliseconds as the Fundamental. For this mathematical ‘trick’ to work, the period has to be 16 milliseconds, just like the Fundamental; consequently ‘m’ and ‘n’ have to be Natural Numbers.

I have shown the ‘hard’ (Integral Calculus) Maths bit as an Equazor DrawFile. You can see that we start with the Pressure Waveform, p = sin(πmt/8) and, by using this algorithm, we are able to extract the Frequency we want by finding the ‘n’ for which this integral returns the value 8. When we have chosen the right ‘n’ then the ‘m’ we want is given by m = n.

If you are into the integration of trigonometrical functions then you will know that the way to integrate $\sin^2(t)$ is by using the identity $\sin^2(t) \equiv (1 - \cos(2t))/2$. When $\sin^2(t)$ is integrated between the limits 0 and 2π it returns the value π.

$$For\ m, n = 0, \ldots +8$$

$$F(n) = \int_{t=0}^{t=16} sin(2\pi nt/16) \times sin(2\pi mt/16)dt$$

$$= 0 \qquad for\ all\ n \neq m$$

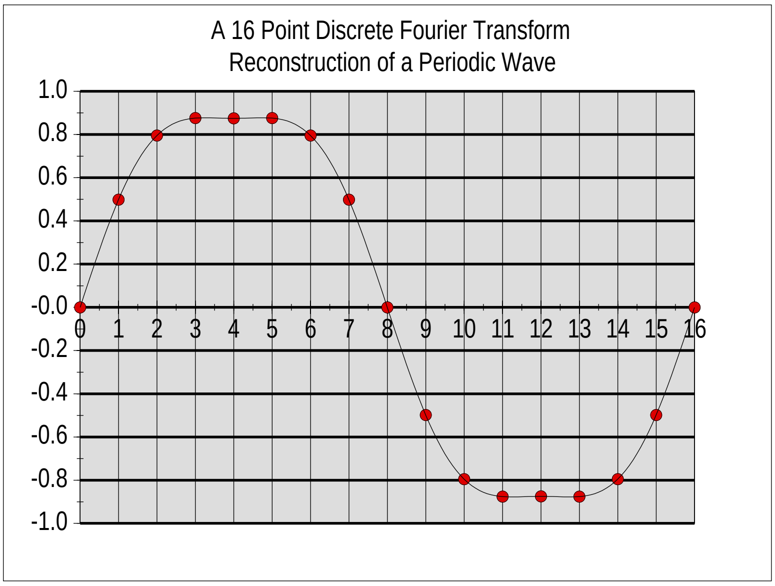
$$F(n) = 8 \qquad when\ n = m$$

More than one tone

Now let us consider our Pressure Waveform to be not a single pure tone but a mixture of two pure tones. For this example I have chosen a Pressure Waveform having the formula p = sin(πt/8) + (1/8)sin(3πt/8). This Pressure waveform is the sum of two sine waves. The second term is the third Harmonic of the Fundamental and to us this third harmonic will sound like a note 2 octaves above the Fundamental with an Amplitude which is 1/8th of the Fundamental waveform.

The second term of the Pressure Waveform formula, p = sin(πt/8) + (1/8)sin(3πt/8) has an Amplitude which is 1/8th of the Amplitude of the Fundamental. As you will see from the graph with the rather long title “A 16 Point Discrete Fourier Transform Reconstruction of a Periodic Wave”, this 3rd Harmonic flattens the peaks of the Waveform.

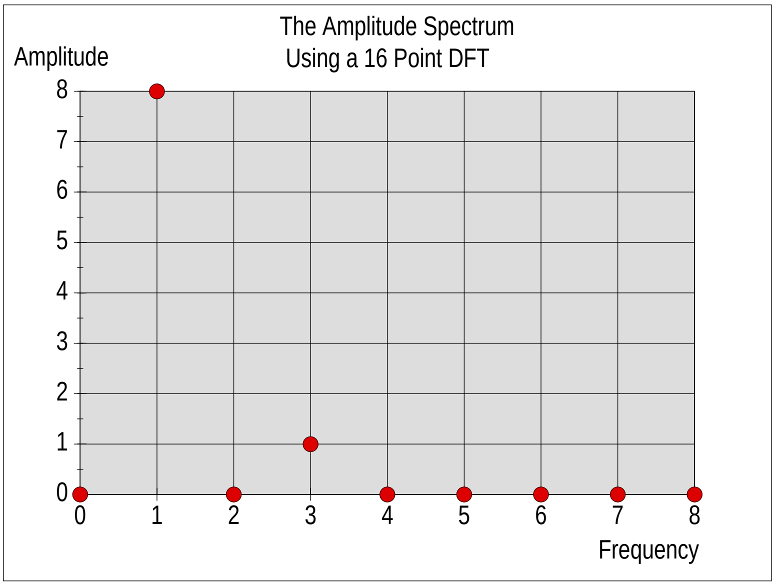
Different musical instruments add different amounts of either odd or even harmonics; it’s these extra Harmonics that gives each musical instrument its unique, distinguishable voice. If we combine two tones which are the 2nd and 3rd harmonics without the Fundamental, then the chord produced is called ‘A Perfect Fifth’. Chords consist of many frequencies often played by different instruments. Using our ears and brain we are able to separate the different frequencies and to distinguish one instrument from another.



Finding the Fourier Transform

What we have to do is to ‘Find the Fourier Transform’ of $p = \sin(\pi t/8) + (1/8)\sin(3\pi t/8)$. The Fourier Transform is the Spectrum of Frequencies contained in the original waveform.

In order to do this and provide a general solution I have constructed a set of spreadsheets which are available from all the usual places. The chart with the title, “The Amplitude Spectrum”, shows the Amplitude of each of the 9 frequencies.



In this chart the Independent Variable is the Frequency; the range is from 0 to 8 cycles per 16 milliseconds. The Dependent Variable is the Amplitude of each Frequency present in the composite Waveform.

The Fundamental has an Amplitude of 8 at the Frequency, m = 1 cycle per 16 milliseconds. The 3rd Harmonic, m = 3, has an Amplitude of 1.

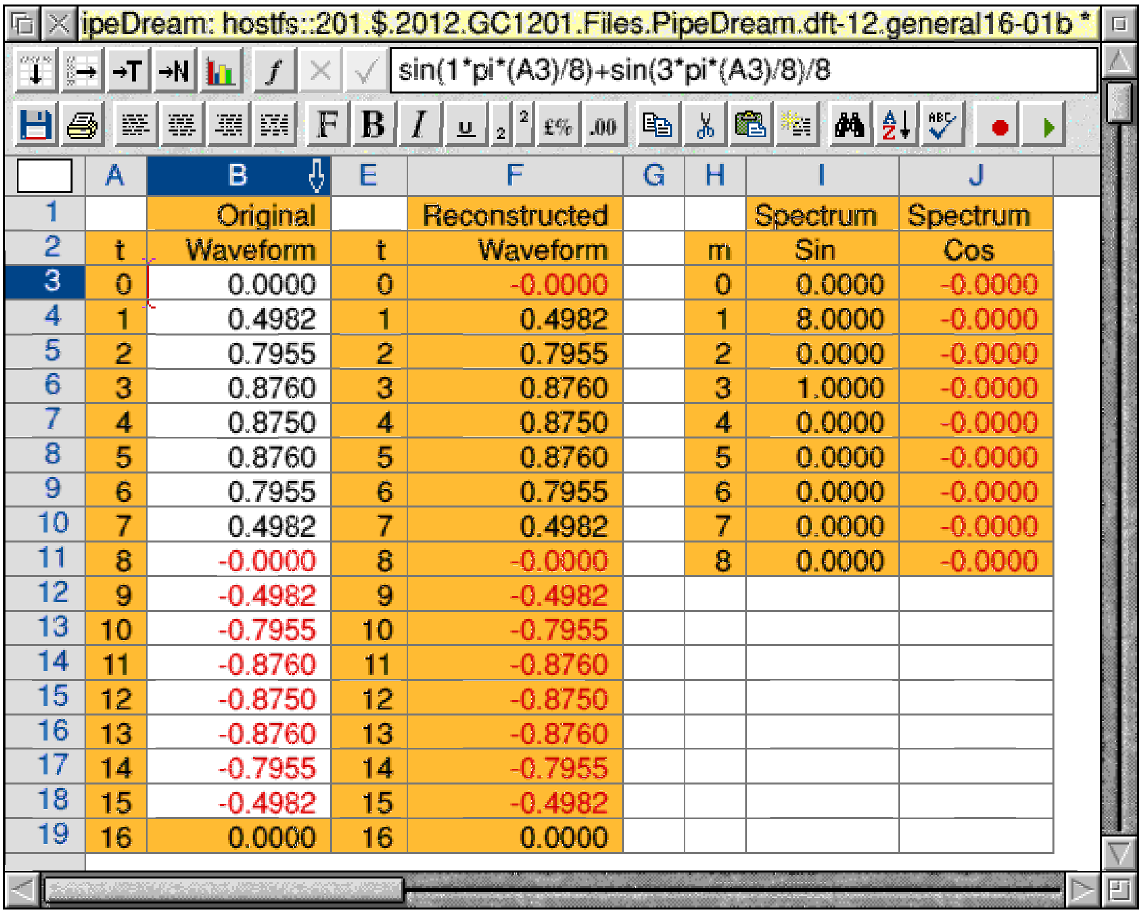
By Finding the Fourier Transform of the Pressure Waveform we have successfully changed our Viewpoint from that of a Time dependent Pressure wave to regarding the sound wave as a Spectrum, a mixture of Frequencies. This process is what our ears and brain do with a composite air Pressure Waveform; they find the Fourier Transform of the sound wave and we hear the mixture of tones. We do not experience the sound as being a Time dependent Waveform but as Music or Speech having different frequencies and their tonal harmonics.

Finding the Fourier Transform of a Waveform is the mathematical equivalent of using a physical piece of kit called a Spectrum Analyser. Our ears act as a Spectrum Analyser.

The ‘General’ Spreadsheet

I have taken a screenshot of the main spreadsheet of the linked set of the 3 spreadsheets which I’ve used to find this Frequency Spectrum. I have called this main spreadsheet ‘general16-01b’. Most of this spreadsheet has protected cells having a yellow brown colour (is this colour called Ochre?). You cannot change the content of those cells without removing the protection. However, you can enter 16 values, any values you choose, into the block B3B18 and, in columns I and J, the Frequency Spectrum of a periodic waveform fitting those 16 points will be returned. In Column F I have reconstructed the original Waveform from the Spectral components. More about this in another article.

The screenshot shown here contains the formula sin(1*pi*A3/8)+sin(3*pi*A3/8)/8 in cell B3; it is replicated down to B18. The spreadsheet as shown in this screenshot calculates the values of p (the Pressure) corresponding to the formula I used for the previous section, namely p = sin(πt/8) + (1/8)sin(3πt/8). Columns I and J contain the Spectrum of this wave.

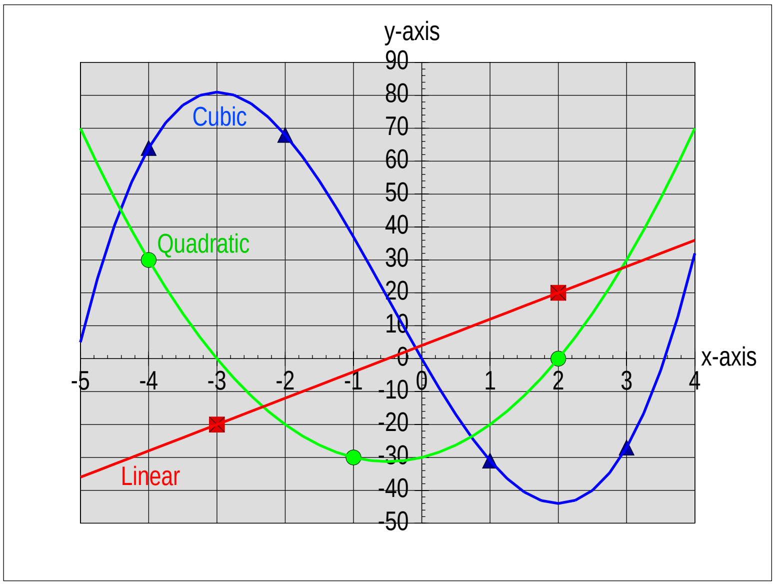


	A	B	E	F	G	H	I	J
1		Original		Reconstructed			Spectrum	Spectrum
2	t	Waveform	t	Waveform		m	Sin	Cos
3	0	0.0000	0	-0.0000		0	0.0000	-0.0000
4	1	0.4982	1	0.4982		1	8.0000	-0.0000
5	2	0.7955	2	0.7955		2	0.0000	-0.0000
6	3	0.8760	3	0.8760		3	1.0000	-0.0000
7	4	0.8750	4	0.8750		4	0.0000	-0.0000
8	5	0.8760	5	0.8760		5	0.0000	-0.0000
9	6	0.7955	6	0.7955		6	0.0000	-0.0000
10	7	0.4982	7	0.4982		7	0.0000	-0.0000
11	8	-0.0000	8	-0.0000		8	0.0000	-0.0000
12	9	-0.4982	9	-0.4982				
13	10	-0.7955	10	-0.7955				
14	11	-0.8760	11	-0.8760				
15	12	-0.8750	12	-0.8750				
16	13	-0.8760	13	-0.8760				
17	14	-0.7955	14	-0.7955				
18	15	-0.4982	15	-0.4982				
19	16	0.0000	16	0.0000				

If you download a copy of this (and the two supporting spreadsheets) and load all three into PipeDream (or Fireworkz) then you will be able to enter your own 16 values into B3B18 and find both the Sine and Cosine components of the Spectrum of your 16 point periodic function. Both Sine and Cosine components are needed for reconstruction of the Amplitude and Phase; more about Amplitude and Phase in another article.

Curve Fitting

In an earlier article, “Curves are Better”, I explained how Matrix methods could be used to fit a polynomial of any order (any power) to a set of points. I used as an example the cubic polynomial shown in the graph. It has the formula $y = 2x^3 + 3x^2 - 36x + 0$; it passes through the four points (−4, 64), (−2, 68), (1,−31) and (3,−27).



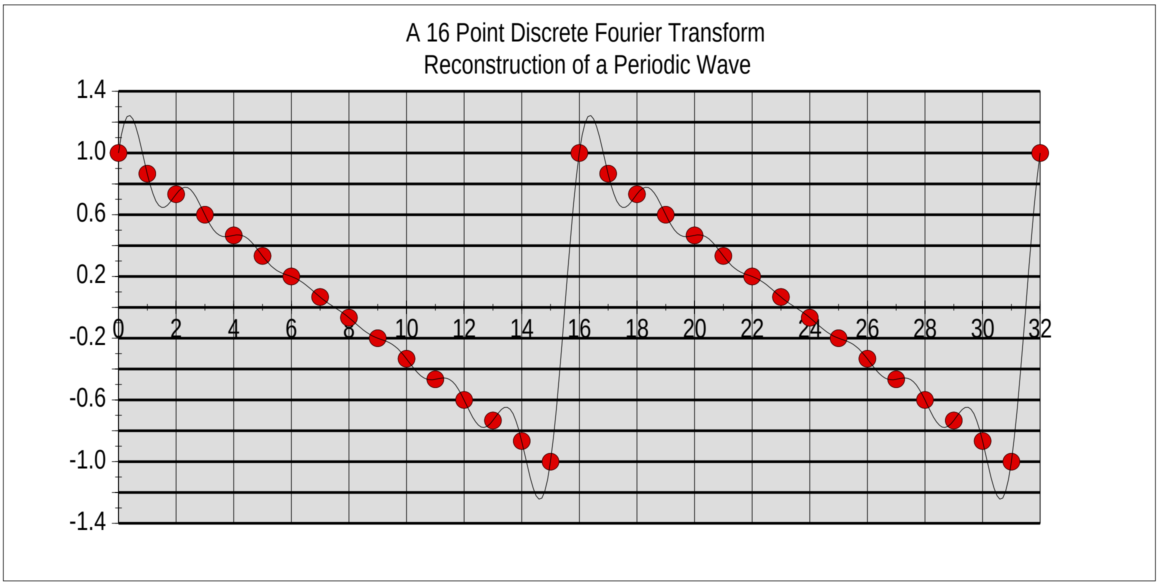
The most serious disadvantage of fitting a polynomial to a set of points is that nearly all polynomials shoot off to infinity once you extrapolate to values of x which are outside the range of given points. Most of the ordinary variables we come across in this world don’t do that! Polynomials are OK for interpolation but they are not appropriate for extrapolation.

I was inspired to write this series of articles about Discrete Fourier Transforms after reading the excellent article by Nicholas Cutler in Archive Vol 23 No 2 called “The Secret Life of Algorithms”. Nicholas’ article referred to an algorithm for calculating Fast Fourier Transforms. The FFT to which he refers is a fast practical method of calculating the Discrete Fourier Transform (the Sine and Cosine Spectral Analysis) of a periodic wave.

Fourier Analysis allows us to create periodic Functions which repeat endlessly even when the values of the independent variable are extended outside the range of given points.

It is usual for the number of points (I chose 16) to be an Integer Power of 2 such as $2^4 = 16$, $2^{10} = 1024$, $2^{14} = 4096$ because it makes the calculations easier. The calculation of a Discrete Fourier Transform is easier because it allows the use of the algorithm known as the Fast Fourier Transform which was described by Nicholas.

I chose 16 points because 16 is small enough to allow the waveform to be analysed and then synthesised easily whilst being large enough to demonstrate all the principles involved in finding a Discrete Fourier Transform. Using 16 points allows us to find a Spectrum but only with frequencies up to the 7th Harmonic. A much larger number of points, for example 1024 points, would allow us to find the Spectrum up to the 511th Harmonic.



If you look at the graph, “Reconstruction of a Periodic Wave” you will see the original 16 red points which I’ve used to create 2 cycles of the Spectrum of frequencies. Also you will see a smooth black line. This is the line showing the ‘in between’ values returned by the analytical extension of the periodic curve using the periodic functions from the Spectrum we have calculated. This reconstruction does pass through all 16 points. However, it doesn’t exactly match the linear (straight line) Ramp Wave of the original 16 points but deviates on either side of that straight line - but it deviates only between the 16 points.

In a future article I shall show you how a 16 point Discrete Fourier Transform recreates an almost square wave which is not as square as the original.

The limitation of using 16 rather than more points is that the Spectrum of frequencies we can use to recreate the waveform is limited to the 7th Harmonic. This limitation can be compared with the limitation of using an 7th power polynomial to find a curve passing through 8 points. The power of the polynomial is one less than the number of points. In the case of the Discrete Fourier Transform the highest harmonic frequency included in the reconstructed waveform is one less than half the number of points - $(16/2 - 1) = 7$.

The reason why the black line deviates from the straight line passing through the 16 points is because we have selected only 16 points. If we forced our black line, the reconstruction of the original waveform, to conform more closely to the straight line then we would need more points and a larger number of harmonics.

You will see that I have included a 17th point at $t = 16$ which is repeated at $t = 32$. This is to show you that the reconstructed waveform is periodic and, unlike the best straight line through the original 16 points, it does not shoot off to infinity outside the range of times for which we have data.

Another way of drawing smooth curves through points is by using Bezier curves. Just as in the case of Polynomials and, in this article, Discrete Fourier Transforms, the actual process of calculating the Parameters of Bezier curves requires the use of matrix methods. Matrix methods, which are built into both PipeDream and Fireworkz, do make the formulae used to compute these Parameters in a spreadsheet, simpler to write and simpler to execute.

Bio-Bit

A very long time ago, in the 1970s, one of Gerald’s inventions, a heat pipe used as a cooking skewer, appeared on Tomorrow’s World and later on Pebble Mill at One. These two 40 year old cooking skewers still work well.