Curves are Better
by Gerald Fitton

Before the Roman came to Rye or out to Severn strode,
The rolling English drunkard made the rolling English Road.

Gilbert Keith Chesterton (1874 - 1936)

The Romans had an unhealthy preference for straight roads. Perhaps this was because, "A straight line is (and was) the shortest distance" for their foot soldiers to walk. Alternatively it could have been because, by using straight lines between their 'points', they were able to draw their maps more accurately. Whatever the reason, as G K Chesterton points out in this poem, the curvaceous roads made by the rolling English drunkard are more attractive in so many different ways than are the unimaginative linear creations of the Roman.

The Greeks knew a lot about curves. Their formal geometry may have extended only to circular arcs but there is no doubt that their appreciation of curves far exceeded their ability to express the beauty of, say, a curvaceous Grecian Urn using any of their mathematics.

Nowadays, thanks to Déscartes and the wonders of Co-ordinate Geometry, we can express all sorts of delightfully delicious curves as mathematical formulae; not only can we find formulae for parabolas and cubics (see the graph) but also for the shape of Marilyn Monroe's silhouette - though the latter formula is rather more complex than the former.

Points and Curves

Last month we concentrated on finding a formula for a straight line through 2 points.
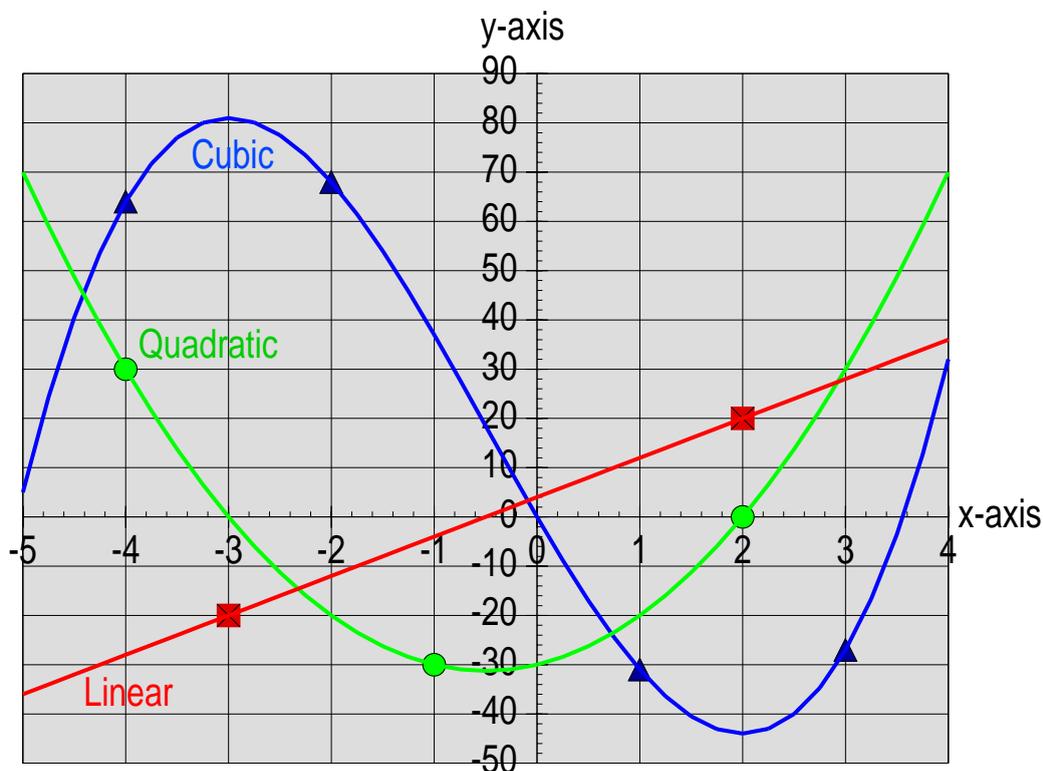
This month we will concentrate our attention on curves. Whether it's a curvaceous Grecian Urn, the silhouette of Marilyn Monroe or the balance and symmetry of a well crafted font, there is something about curves that we find much more attractive than straight lines.

In the graph you will see 2 curves, the Quadratic and the Cubic curves. The 'best' curve through 3 points is the Quadratic and the 'best' curve through 4 points is the Cubic curve.

A Quadratic Needs Three Points

To specify a unique quadratic curve we need 3 points. In my quadratic example these 3 points are (-4, +30), (-1, -30) and (+2, 0). The 'do it backwards' problem is to find the 'a', 'b' and 'c' parameters of the quadratic function, $y = a*x^2 + b*x + c$, which allows us to evaluate y for any and every possible value of x.

In order to find these unique values of a, b and c we substitute the 3 values of x and y in the generalised version of the function to yield the 3 simultaneous equations:

$a*(-4)^2 + b*(-4) + c = +30$
$a*(-1)^2 + b*(-1) + c = -30$
$a*(+2)^2 + b*(+2) + c = 0$

Next we convert these 3 simultaneous equations into the matrix format, $M*A = Y$, where A is the column matrix of the 3 unknowns, a, b and c, M is the matrix of the coefficients (these depend on the values of x) and Y is the column matrix of the 3 values of y.

As a Fireworkz spreadsheet the matrix format of the problem looks like this:

The block d4d6 contains the values of the variable x, c4c6 contains the values of x^2 and the block g4g6 contains the corresponding values of the variable y.

We solve these 3 simultaneous equations for a, b and c using matrix algebra.

Starting with $M*A = Y$ we premultiply both sides by the inverse of M (conventionally given the symbol M^{-1}). This gives us $M^{-1}*M*A = M^{-1}*Y$.

The reason we premultiply by M^{-1} is because $M^{-1}*M = I$, where I is the Identity Matrix.

I described this Identity Matrix in detail in Archive 22:5 pp27. Its most important property is that, just like the integer 1 in the multiplication operation $1*x = x$, carrying out the corresponding matrix operation $I*A = A$ gives us A. The solution to our 'do it backwards' problem is A, the column matrix of the parameters a, b and c.

Summarising, the matrix equation $M^{-1}*M*A = M^{-1}*Y$ becomes $A = M^{-1}*Y$. The block g9g11 (which is the matrix A) contains a, b and c.

Translating this A matrix back to individual numbers gives us a = +5, b = +5 and c = -30.

We insert these values for the parameters a, b and c back into the general quadratic (for any parabola), $y = a*x^2 + b*x + c$ to give us $y = 5*x^2 + 5*x - 30$. We have found the unique function relating y and x for the parabola passing through (-4, +30), (-1, -30) and (+2, 0).

The Clever Bit

Of course the 'clever bit' is finding M^{-1}. How? Don't panic! Fireworkz will do it for you!

The formula in cell c9 is set_value(c9e11, m_inverse(c4e6)); the 'clever bit' is carried out using the matrix function m_inverse(matrix). The set_value(destination,source) function expands the matrix (held as a 3x3 array in the single cell c9) into the block c9e11.

There is only one other matrix formula in the spreadsheet and that is in cell g9. The formula in g9 is set_value(g9g11, m_mult(c9e11,g4g6)); it is the matrix multiplication which executes $M^{-1}*Y$ to return A in the block g9g11.

Enter Only the Values of x and y

The block d4d6 contains the values of x; these have to be entered into the spreadsheet manually. The values in c4c6 are the values of x^2. These do not have to be entered manually because Fireworkz can do it automatically. The formula $d4^2$ has to be entered into c4 and then replicated downwards to c6 generating the squares of all the x values.

The block g4g6 contains the corresponding values of y; these have to entered manually. This Fireworkz spreadsheet calculates the values of the parameters a, b and c automatically and returns them into the block g9g11.

Once the spreadsheet has been set up then it will find a, b and c for the quadratic passing through any 3 points; the x and y values are entered into d4d6 and g4g6 respectively.

Drawing the Graph

I used the PipeDream file poly-01 to draw the graphs shown at the beginning of this article. The file poly-02 is used to insert the 9 points on this graph. These files are available on the website. The chart is the file poly_C; it uses both poly-01 and poly-02.

A Cubic Needs Four Points

A unique cubic curve curve requires 4 points to be specified. In my example, these 4 points are (-4, +64), (-2, +68), (+1, -31) and (+3, -27). The 'do it backwards' problem is to find the 'a', 'b', 'c' and 'd' parameters of the cubic function, $y = a*x^3 + b*x^2 + c*x + d$. This cubic formula can be used to evaluate y for any and every possible value of x.

I'm sure that most of you won't need me to repeat all the boring detail about there being 4 simultaneous equations in a, b, c and d but will be content to see me jumping straight to the matrix format, $M*A = Y$. Here is a screenshot of the spreadsheet.

This matrix technique can be extended to 4 points. The matrix equation, $M*A = Y$, is identical however many points you have. The solution to this matrix equation is identical to the matrix solution of the quadratic function, $A = M^{-1}*Y$, the only difference is that for the cubic, the 3x3 matrices (of the quadratic problem) are replaced with 4x4 matrices.

This matrix format of the problem can be extended to more than 4 points.

Inverting a 4x4

I guess that with all the snow we've had lately it wouldn't be too difficult to invert a 4x4 vehicle - but it's not vehicles that I'm writing about. It's 4x4 matrices.

PipeDream and Fireworkz have built in functions for inverting 2x2 and 3x3 matrices but there is no built in function for inverting 4x4 (and larger) matrices. This is where Bob Ardler has come to our rescue with a ready made custom function for inverting a matrix of any computable size; the only limitation is the capacity of the computer being used.

Within his elegantly crafted custom function, c_MatOps, you will find many interesting matrix functions; the one relevant to our problem is [c_MatOps]inverse(matrix).

The Matrix Solution

The cell b9 of the spreadsheet poly-05 shown in the screenshot above contains the formula: set_value(b9e12,[c_MatOps]inverse(b4e7). It is this custom function of Bob's which inverts the 4x4 matrix of the coefficients (of a, b, c and d) contained in the block b4e7.

The 'do it backwards' solution is the block g9g12, a = 2, b = 3, c = -36 and d = 0.

The formula for the cubic is y = 2*x^3 + 3*x^2 - 36*x + 0.

The Chart

The chart shown at the beginning of this article is the PipeDream Chart file poly_C. The formula for the cubic is evaluated for many points in column D of the file poly-01; the Scattergraph option of PipeDream is used to draw the lines on the chart poly_C.

I used PipeDream rather than Fireworkz for the chart because PipeDream has a much more flexible charting package than has Fireworkz. Using PipeDream I was able to add and position precisely all the labels and captions which you can see on this chart; it would have been much harder to do this using Fireworkz.

Even More Points

This same matrix solution, $A = M^{-1}Y$, to finding the 'best' polynomial, can be extended to many more points. You will need extra columns for x^4, x^5, etc, but these values can be calculated automatically by Fireworkz. Bob Ardler's custom function will handle matrices much larger than 4x4. I've tried it on a 20x20 and it works!

Back in the days of long ago it took a team of 20 girls up to a week (working for me on aeroplane designs) to invert a 20x20 matrix; Bob's custom function is much faster! When it comes to calculating formulae for curves, a computer is more efficient than 20 girls - but, you must agree, not better in every curvaceous way! I must add that it was through my need for calculating girls that I met my wife, Jill; she was by far the most curvaceous of those 20 clever girls.

A Better Straight Line

You may not want to draw a line in such a way that it wriggles its high order polynomial way through every point. What you may want to do is to draw your 'best straight line' in such a way that it narrowly misses those points which are not quite on the straight line.

In another article I will describe the matrix technique for drawing not only a straight line which narrowly misses a set of points but the also best parabola, the best cubic and even the best quartic which narrowly misses points which are not quite on those polynomials.

But that will have to be another article for another time ...

Box-Out - G K Chesterton

My friends, we will not go again or ape an ancient rage,
Or stretch the folly of our youth to be the shame of age,
But walk with clearer eyes and ears this path that wandereth,
And see undrugged in evening light the decent inn of death;
For there is good news yet to hear and fine things to be seen,
Before we go to Paradise by way of Kensal Green.

Chesterton was a big man. He stood 6 feet 4 inches tall and weighed over 20 stone. As well as writing poems, books, essays and plays he wrote several articles for the Encyclopaedia Britannica. His 'day job' was as a prolific newspaper columnist.
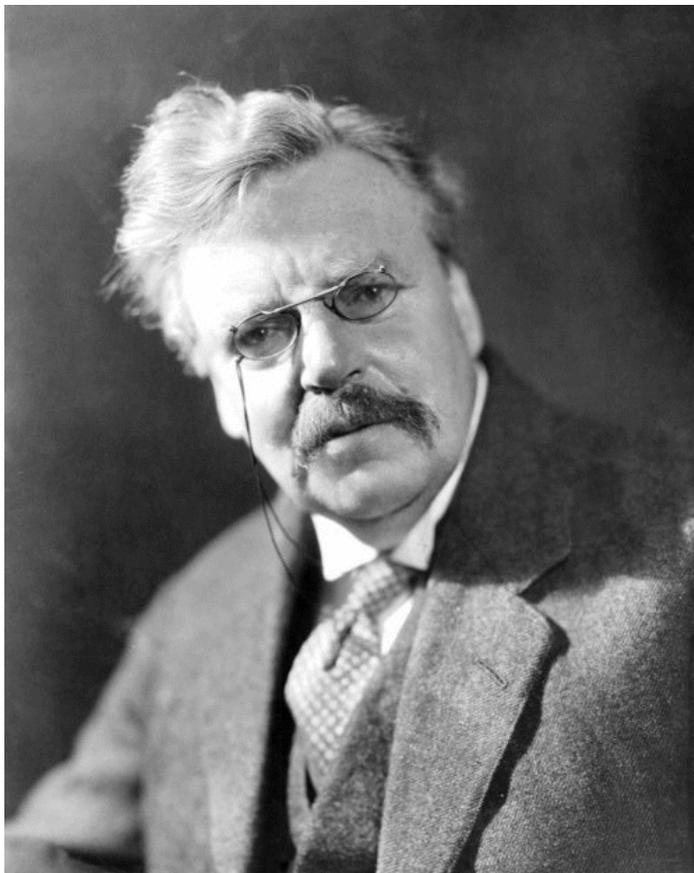
He and his wife were devoted. Whether it was absent mindedness or too many mugs of ale, all too often he would find himself in some distant location such as, "Glastonbury by way of Goodwin Sands". Not knowing where he should be, he would send a telegram to his wife, "Where should I be?" Her one word reply, "Home!"

Although, during the follies of our youth, we might have followed his Rolling English Road to, "Birmingham by way of Beachy Head" or even to, "Bannockburn by way of Brighton Pier", it is with the clearer eyes of maturity that we can appreciate those many fine things both to see and to do which we missed as we went down a curvaceous English lane with ale mugs in our hands. Perhaps we wouldn't see nor experience them now if we had not enjoyed fully. "the folly of our youth"?

Box-Out - Bézier Curves
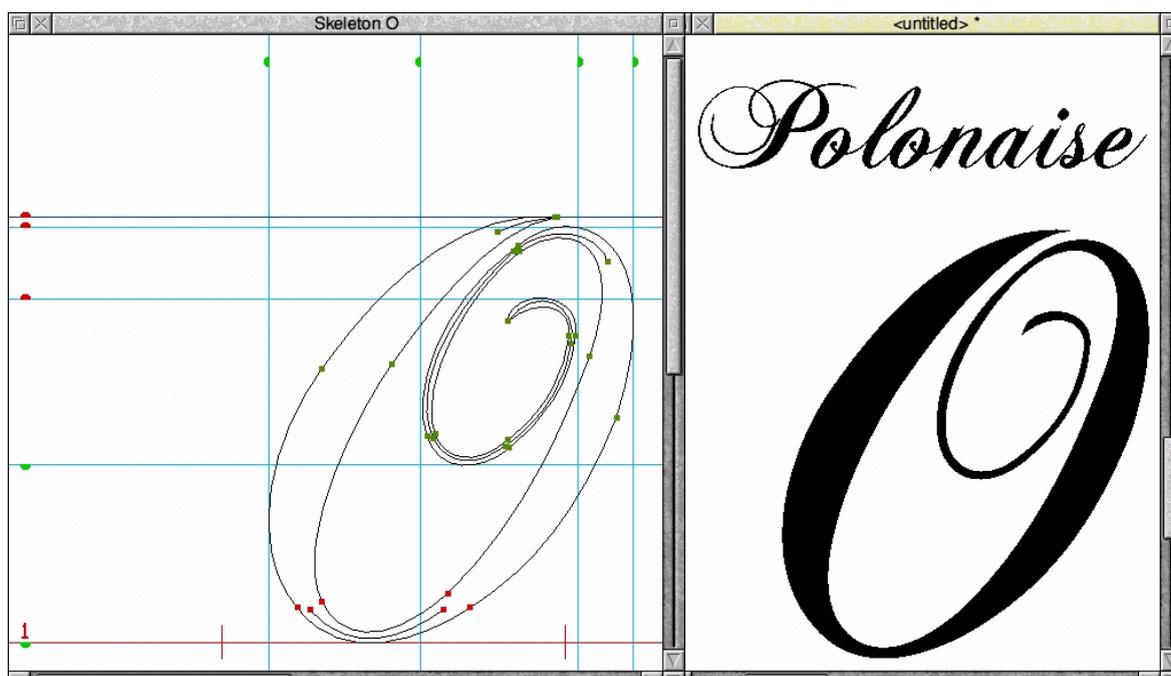
Although, in this article I have concentrated on the mathematics of drawing polynomial curves through a few points, there are applications for which Bézier Curves are more suited. The shapes of characters in a font (these are called glyphs) is one such application of Bézier Curves.

This glyph for the letter 'O' is drawn by the Font Manager from only a couple of dozen control points, but, as you can see, it has a curvaceous beauty which only a Master (or Mistress) of font design could bestow upon it.

I shall write about the Mathematics of Bézier Curves (and their applications to, for example, the design of fonts) in another article on another day.

Bio-Bit

Gerald maintains that as he gets older his memory is improving. He insists that he can remember perfectly the maths (eg matrix algebra) he was taught over half a century ago. "Now, where did I put those car keys? Oh! They're in my hand!"
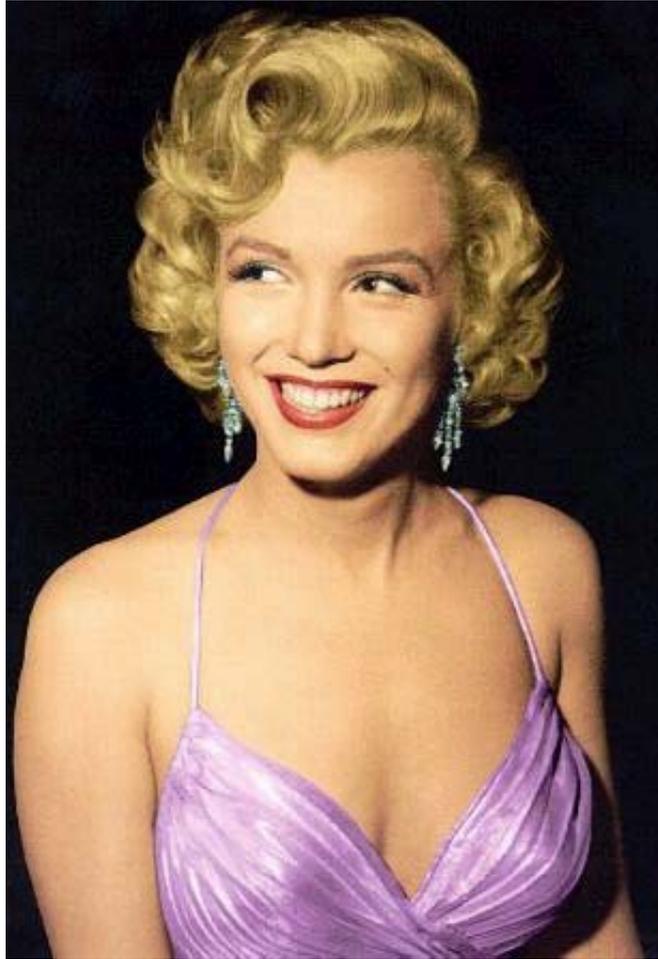
The Curvaceous Marilyn Monroe whose Bézier Curves have appeared on many websites.