Lateral Thinking
by Gerald Fitton

If the way forward is impassable
Then find a way around the obstruction

Lateral Thinking (Unpublished)
Gerald (1932 -)

Let me make your heart sink.

Alice has four more sticky sweets than Bob.
Together they have twenty sticky sweets.
How many sticky sweets does Alice have?

The Horrible Sticky Sweet Problem

I recall asking a group of students this question. The first thing they wanted to know was why Alice should have more than Bob. Was it because she was prettier than Bob? "No!" I replied. Both Alice and Bob had been give the same number of sticky sweets by their unknown benefactor but Bob had given four of his to his girlfriend, Cheryl, who was even prettier than Alice. That's all (for now) of Cheryl's part in the Saga of the Sticky Sweets.

The first student to come up with the correct answer used the following reasoning:

Share out the sticky sweets equally between Alice and Bob, ten each. Then Bob gives Alice one of his ten so that Alice now has two more than Bob (11 - 9 = 2). This isn't a large enough difference; so Bob gives another of his sticky sweets to Alice. Yippee! Now Alice has four more of these absolutely horrible sticky sweets than Bob (12 - 8 = 4). Alice has the original ten plus two given her by Bob to make twelve in all! My student had solved the Horrible Sticky Sweet Problem in a manner totally unexpected by me!

Simultaneous Equations

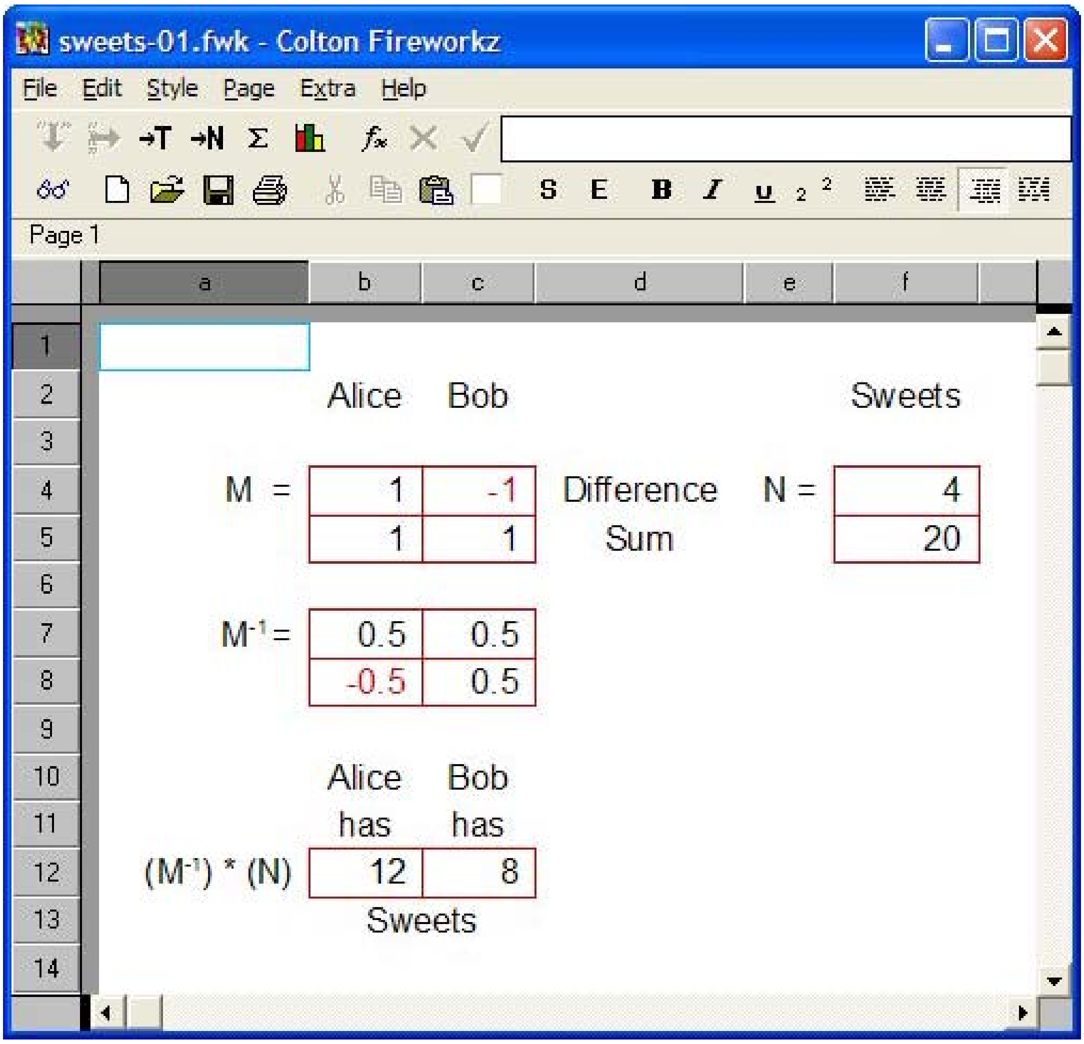
The mathematicians, and even those erstwhile mathematicians, among you will have recognised that this problem is a simple (!) example of a pair of simultaneous linear equations with two unknowns. I shall use the letter “a” for the number of sweets held by Alice; I’ll make you guess at the meaning I’ve given to “b” when I write:

$$a - b = 4$$
$$a + b = 20$$

There are many ways of solving a pair of simultaneous linear equations such as this; methods include “systematic elimination” (eg add the equations together to eliminate Bob) or by drawing straight lines on graph paper. One of my favourite methods for harder, non liner equations (eg polynomials or trigonometrical equations) is a numerical method which homes in on the answer to any degree of accuracy which you wish to choose - but not now!

For simultaneous linear equations there is a wonderful semi-automatic method which makes use of matrix multiplication. Matrix multiplication is built into PipeDream and Fireworkz. It is this magnificent matrix method which I shall describe this month.

The Alice and Bob Matrix



Have a look at the screenshot.

In column ‘f’ you will see the number 4 in f4 and 20 in f5. I have written “Difference” and “Sum” in d4 and d5 respectively so that, without me saying any more, you will realise the significance of these two numbers. I have called this column matrix, (4;20), as ‘N’.

Let me explain my use of a semicolon in (4;20). This notation follows the notation used by both PipeDream and Fireworkz for entering a numeric matrix into the formula line. A semi colon can be thought of as a ‘Next Row’ data separator so that the ‘4’ of the (4;20) is on the top row of the matrix and the ‘20’ is on the second row of the matrix.

In rows 4 to 8 you will see a couple of 2x2 matrices called ‘M’ and ‘M^{-1}’.

Near the bottom on the left you will see in column ‘b’ that “Alice has 12 Sweets”; in column ‘c’ you’ll see “Bob has 8 Sweets”. Being observant (as well as highly intelligent) you will have noticed that it is the matrix multiplication of (M^{-1}) * (N) that produces the row matrix (12, 8), the number of sticky sweets held by Alice and Bob respectively.

The comma in (12,8) can be thought of as a ‘Tab’ data separator so that the ‘12’ is in the first column an the ‘8’ is in the second column of the transposed matrix of the unknowns.

The Matrix M

The two simultaneous equations ...

a - b = 4
a + b = 20

... can be rewritten in matrix format as (M) * (A) = (N) where M is the matrix M shown in the screenshot and A is the column matrix of the unknowns, (a;b). Note the use of the semi colon as an ‘end of row’ data separator; using that notation the right hand side of the pair of equations becomes the column matrix (N) = (4;20).

Solving an equation

I am sure that you are all aware of the method of solving a simple equation such as 4x=20. The method is to divide both sides of the equation 4x=20 by 4. The result is (4/4)x=(20/4) which can be simplified to 1x=5 or, with the ‘1’ as silent as the grave, we can write x=5.

This process works because we know that if we divide 4 by 4 then the answer is 1; also we know that ‘x’ multiplied by ‘1’ is still the same old ‘x’.

Having written (M) * (A) = (N) what we would dearly love to do is to divide both sides of this matrix equation by the (M) to give us (A) = (N)/(M). I assure you that if you could do this then the answer would be (A)=(8;12) just as it is shown in the screenshot.

The ‘problem’ which faces you is that, although I have told you quite a bit about matrix multiplication I have not told you how to do matrix division. Is that because I am a ‘meanie’ and want to keep division to myself? No I assure you it is not. Read on.

Matrix Division

If you look very closely at the Spreadsheet Function Guide for Fireworkz (or study the PipeDream manual) you will find all sorts of matrix functions including the m_mult(,) (matrix multiply) function but nowhere will you see anything which looks even vaguely like m_divide (,). Why is this? Were Colton Software (or their programmers) so totally incompetent that they were unable to program this highly desirable function? Perhaps what is needed is a simple (!) custom function to do matrix division. Let's get out a text book and see what we can find - and then ask Gerald to write a custom function.

I have here, in this room with my computer, a wonderfully comprehensive variety of mathematics text books. I assure you that an algorithm for matrix division does not appear in any of them. The reason is that a matrix operation 'divide' does not exist. Indeed, the situation is more serious than "does not exist" implies. Matrix division cannot exist.

So since matrix division is impossible (honest, it can't be done, not even by me!) we need to think again. It is no use beating our heads against a brick wall bemoaning the impossible; what we need here is a bit of lateral thinking. We need to find a way around this impenetrable barrier labelled, "Matrix Division is Impossible".

The inverse

Let's look again at the simple equation 4x=20 but we'll look at what we did in a different way. I suggested that we should divide both sides of this equation by '4' but equally I could have suggested that we multiply by 0.25 or 1/4 if you prefer. We know that we have a matrix multiplication function, so using matrix multiplication with something which does the equivalent for matrices as the (1/4) does for 4x=20, sounds like a promising approach.

Where did the '4' come from in the (1/4)? We chose (1/4) as our multiplier (for both sides of the equation) because we know that (1/4) * (4) = 1 and because 1 * x = x.

The inverse of the multiplication process is called division. We can't divide matrices but we can multiply. The inverse of 4 is (1/4). By this statement I mean that when we do the multiplication (1/4) * (4) we get that 'magic' number '1'. As a by-the-way, the number '+1' is its own inverse because (+1) * (+1) = (+1).

The Identity Matrix

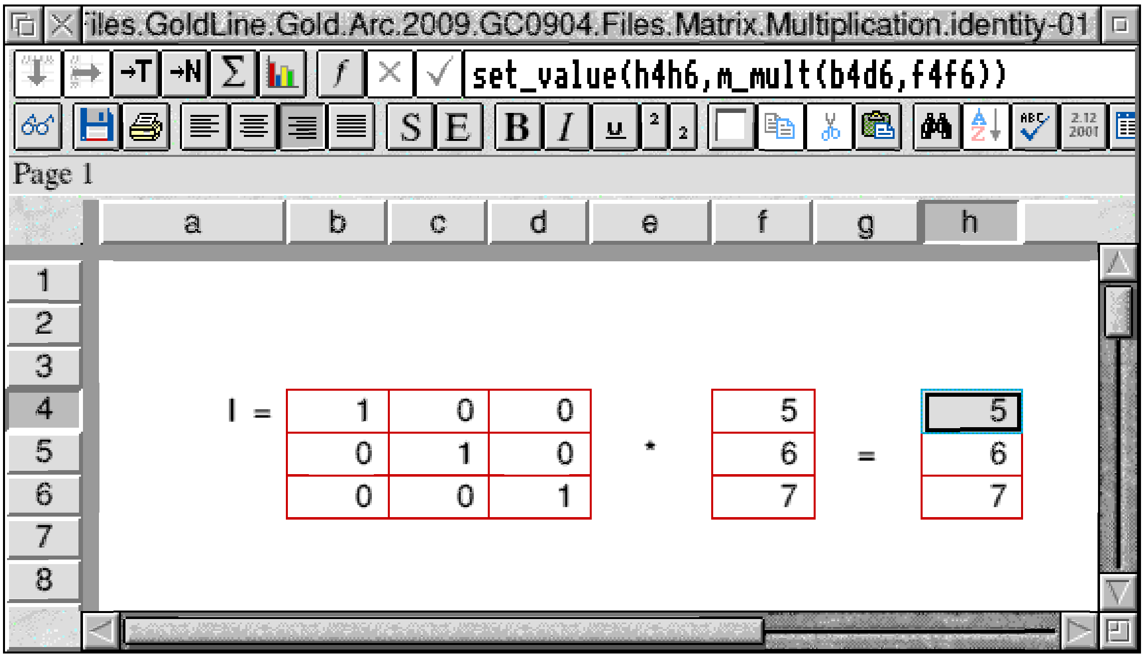
I'm going to hit you with a definition! I shall define an identity matrix called (I).

The Identity Matrix, which I am calling (I), is a matrix such that if you multiply another matrix, let's say matrix (N), by this identity matrix, (I), then you get the same matrix (N) that you started with.

Using matrix notation this multiplication looks like this: (I)*(N) = (N). You can compare this matrix equation with the numeric equation: 1 * x = x.

A good question is, "What does this Identity Matrix look like? What values does it have?"

Rather than try to explain it in words here is a screenshot showing an example of an Identity Matrix, (I).



The Identity Matrix, like the 3x3 example in b4d6, is always a square matrix, but it can be any size. It has the number '1' in every cell of the diagonal and the number '0' elsewhere.

If you multiply any matrix having three rows (and as many columns as you like) by the 3x3 matrix shown in the screenshot then the answer will be the matrix you started with.

In the screenshot I have used matrix multiplication to pre-multiply the column matrix (5;6;7) by (I) and the answer is (5;6;7), the matrix we started with.

The Inverse Matrix

So where do we go from here?

Again I'll start with a definition.

If (M) is any square matrix (2x2 or 3x3 or any matrix with the same number of rows as it has columns) then its inverse, which we'll call (M^{-1}), is such that (M^{-1})*(M) = I. The result of pre-multiplying (M) by its inverse, (M^{-1}), is the Identity Matrix, (I).

The clever bit

We have the matrix equation (M)*(A) = (N).

Let me remind you of the simultaneous (Horrible Sticky Sweets) equations this represents:

a - b = 4
a + b = 20

This pair of simultaneous equations can be written in matrix form with
(M) = (+1,-1;+1,+1) and (N) = (4;20) as (M)*(A) = (N).
You can see this if you go back to the first screenshot.

The next step, which I shall skip over for now, is to find (M^{-1}).
The answer, I assure you, is that (M^{-1}) is the matrix (+0.5,+0.5;-0.5,+0.5).

If we pre-multiply both sides of (M)*(A) = (N) by (M^{-1}) then we get:

(M^{-1})*(M)*(A) = (M^{-1})*(N)

Now we know that (M^{-1})*(M) = (I) and that (I)*(A) = (A).

Consequently the left hand side collapses to (A) and we find that (A) = (M^{-1})*(N).

This is the matrix calculation which is in row 12 of the Alice and Bob screenshot.

The difficult bit

If you have followed me up to here then you will realise that the easy bit is executing the matrix multiplication (M^{-1})*(N) (in row 12) to find (A). The difficult bit is finding (M^{-1}).

It is here that Colton Software and Bob Ardler (see below) come to our rescue.

For 2x2 and 3x3 matrices both PipeDream and Fireworkz contain the function m_inverse(matrix) which returns the inverse of any 2x2 or 3x3 matrix. It is this matrix inversion function (together with the set_value(destination,source) function) which I have entered in cell ‘b7’ of the ‘Alice and Bob’ spreadsheet (the first screenshot) to expand (M^{-1}) into the range ‘b7c8’.

Summary

To solve a set of ‘n’ simultaneous linear equations we first convert the set into matrix format so that it becomes (M)*(A) = (N) where the matrix (M) is a square matrix of dimension ‘m’ by ‘m’ and the matrix (N) is a column matrix having ‘m’ components.

Next we find (M^{-1}) - easier said than done. There are computers which are dedicated to doing this for huge sets of equations.

Finally we execute the multiplication (M^{-1})*(N) to find the value of (A).

In my youth one of my first office jobs was to do sums on the stability and control of fighter aeroplanes; I had to invert 20x20 matrices (in the days before computers) in order to find out if the aeroplane being designed could fly ‘hands off’!

It took my team of 20 girls a couple of days to do these calculations! Of course every calculation was done twice so that errors could be detected.

Acknowledgement

My thanks go to Bob Ardler who has provided a well written Custom Function returning the inverse of matrices much larger than 3x3. This custom function works with PipeDream, Fireworkz for RISC OS and Fireworkz for Windows. You will find it on the Archive monthly disc and elsewhere.

Bio-Pic

Last month Gerald's Friends, Rex and Karen Palmer took Gerald to see Fidelio at Glyndebourne. The usual dress is black ties. The ladies in their long (and some short) evening dresses look wonderful. He was treated royally to a wonderful weekend.

Fidelio is a good comic opera and the 'band' played well!

Box-Out

The Laws of Physics as proposed by Newton, Maxwell and others of that era regard space and time as continuous. At school and college I was taught Differentiation, Integration and Complex Regular Functions. It is this branch of mathematics that rules in this continuous but Deterministic model of the Universe.

Werner Karl Heisenberg (1901 - 1976) together with Max Born (1882 - 1970) and the Mathematician Pascual Jordan (1902 - 1980) formulated the Matrix Mechanics representation of Quantum Mechanics.

Matrix Mechanics is the ‘discrete’ version of Quantum Mechanics (requiring the solution of Matrix Equations) as opposed to the better known ‘continuous variables’ formulation of Erwin Shrodinger in his famous Wave Equation. This Wave Equation was constructed in 1926 after Jordan’s formulation of the Matrix Mechanics model.

Matrix Mechanics is an algebra of quantum observables - these observables are the only things we can observe! The continuous variables of Shrodinger are probabilities and are not observable! Many Physicists question the existence of these non observable probabilities but yet they continue to construct models which require them to have some sort of real existence.

It was partly because Physicists couldn’t get to grips with the discrete (matrix) mathematics (developed by Jordan) that they preferred to use the continuous (differentiation, integration, etc) mathematics of the Shrodinger formulation. Even now the emphasis is still on the study of the continuous model of Shrodinger rather than the discrete, genuinely observable observables of Jordan’s Matrix Mechanics model.

On a historical note, it didn’t help the promotion of Jordan’s discrete mathematics that he joined the Nazi party in 1933 and became a ‘brown shirt’, indeed, he lost his chance of a Nobel Prize (he was sponsored by Einstein) because of this!

In the Quantum Physics of Jordan, Heisenberg, et al, the microscopic universe is regarded as discontinuous; even space and time have discrete values rather than being smoothly continuous. It is the mathematics of Eigen values and Matrix Algebra which rule in this non Deterministic, Uncertain Quantum world - whereas the mathematics of continuous functions (differentiation etc) implies that the Universe is totally Deterministic through the smoothly varying continuum of space and time.

Of course we like to believe that we do not live in a Deterministic Universe; it is Quantum Mechanics which allows us to rationalise our belief that we have Freewill.

On a Mathematical note, Shrodinger’s Wave Equation can be transformed into the Heisenberg/Jordan Matrix Mechanics and vice versa. Either can be used to solve practical problems but Matrix Mechanics, once understood, provides a more believable physical model than attempting to conceive in some way the non observable variables (they are just probabilities) of the Shrodinger Wave model.

The continuous nature of space and time is drummed into us at school. The very idea that the Physical Universe might consist of discrete states is so foreign to us that it is not part of the way we ‘feel’ it works. A consequence of this historical fact is that when I mention calculus to most people they will have heard of it and know that it is about smooth changes of state - but if I try to discuss matrix algebra with its discrete variables then this is considered to be irrelevant to anything in real life.

Nowadays powerful computers work day and night solving simultaneous equations having hundreds of unknowns by inverting matrices in an attempt to gain a deeper understanding of this non deterministic, wonderfully uncertain Quantum World in which we can all believe that we enjoy Freewill.

Below are pictures of Heisenberg (top left), Born (top right) and Jordan (bottom).