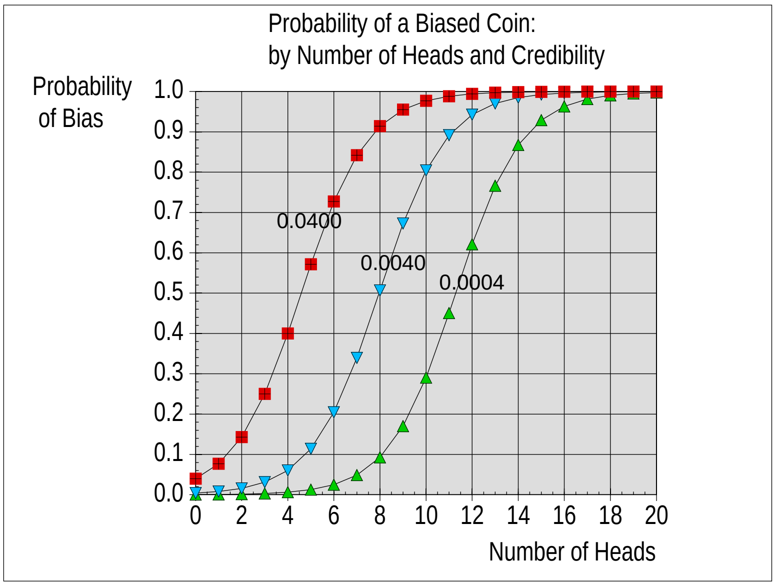
Gerald's Column
by Gerald Fitton

Last month I introduced you to the concept of 'a priori' probability and promised that I would show you how to construct the spreadsheet for my credibility graph.

I suggested that you are more willing to give up on the missing bus after half an hour than you are to suspect some kind of magic trick after five heads in a row—even though the evidence implies that the probabilities are about the same (96% & 97%). I promised to explain you how you might rationalise and even quantify your prejudice.

It's Incredible

Have another look at last month's chart.

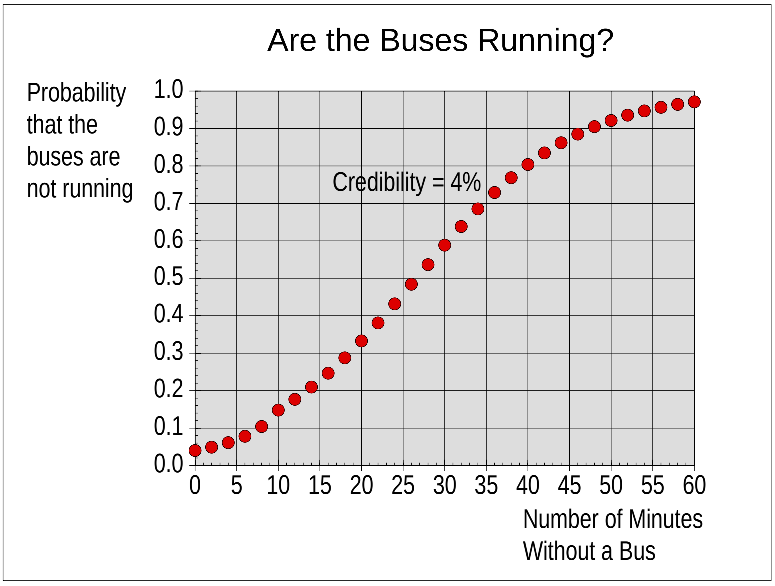


Before we start I would like you to ask yourself this question: "How many heads in a row did it take before you became suspicious?" Please try not to use the 20:20 vision of hindsight but think back to the previous articles in this series and decide on the number of heads in a row it took before you began to suspect some hanky-panky was going on.

From the graph you will see that the Credibility = 4% line reaches 50% probability after five consecutive heads so, if you became suspicious after five consecutive heads then your 'a priori' assessment of my honesty is the complement of 4% namely 96%. Let me put this another way. When I started recounting this tale you could have quantified, albeit unconsciously, the probability of me setting out to deceive you as 4%.

I have used a similar spreadsheet to generate a similar chart for the missing bus. The values of the 'a priori' probabilities are a little different from the ones I chose last month:

Probability that the buses run on time = 30%
Probability that the buses arrive at random with a mean time of ten minutes = 66%
Probability that neither of these is true = 4%



The parameter which I have called Credibility is the same, 4%, as it is for the 4% curve in the flippin' coin chart. You will see that the probability that there isn't going to be a bus, reaches the 50% value at about 26 or 27 minutes.

From the correspondence I have received there are only a few of you who thought that five heads in a row was unusual enough to worry about. Contrast this with my straw poll for the missing bus. Nobody would have waited more than half an hour before reverting to 'Plan B'. The analysis of 'a prori' probability which I have condensed into the two graphs shown above indicates to me that you have more 'faith' in coins being unbiased than buses being cancelled. Let me rephrase this. Whilst your initial doubts about the coin (or me) are less than 4%, your initial doubts about the bus are higher than 4%.

The Flippin' Coin

I would like to give credit to Duncan Breckels for having the courage of his convictions and coming straight out with "You are using a double headed coin!". Duncan suggested that it may have been supplied to me by a company specialising in Educational Resources. It wasn't (I had to devise my own lectures); I inherited the coin from my grandfather.

Back in the early part of the last century my grandfather ran an engineering works in the Manchester area. It was he who made the double headed half crown. I don't know the date because there are no dates on the head side of a coin but it was pre 1920 because the material is sterling silver (a 90% silver alloy). He made it out of two similar half crowns. He turned (on a lathe) the inside of one coin and a matching outside for the other coin. The two parts were 'shrink fitted' using a small amount of heat and one of the works presses.

After his wife died he spent much of his time with me. I suspect that taking me out for the same walks that the three of us went on when she was alive brought her back to him. It was on one of these 'educational' walks that he first introduced me slowly, over a period of about an hour, to this flippin' coin. Many years later, when I was about 19, he gave me the coin and said that he hoped that when I looked at it, its unusual feature would remind me of him. That was fifty years ago and yes, it does remind me of him.

Using up your chances

Before I explain the spreadsheet let me describe the analysis. I shall choose 4% credibility for my worked example. I have chosen 4% rather than the 5% (1:20) value beloved by scientists because I can divide 96 by two five times and still have an easy-to-use integer. These probabilities, the 96% and the 4%, are ‘a priori’ probabilities. I flip the coin and record the results. Applying the outcome of these flippin’ experiments the ‘a priori’ probabilities are modified to become ‘a posteriori’ probabilities.

The experiment consists of throwing the coin many times. As all of us know now, but only because I told you of my grandfather, it comes down heads every time. Cast your mind back to the time when I first told you about the coin. I asked: “What is your best guess at the probability of the fourth throw returning the dreaded ‘Head’?” You had a choice:

(a) The probability is still 50% (the coin doesn’t know what happened on previous throws)
(b) The probability is less than 50% (it’s about time our luck changed)
(c) The probability is greater than 50% (it’s got stuck in a rut).

With ‘a posteriori’ hindsight the correct answer is (c).

Suppose that what we are looking for is a tail. The 96% figure is our ‘a priori’ estimate of the probability that if we flip the coin enough times then we’ll eventually get a tail.

After the first head comes up 50% of our 96% has been ‘used up’ leaving us with 48% plus our original 4%. We have 52% of our original 100% left.

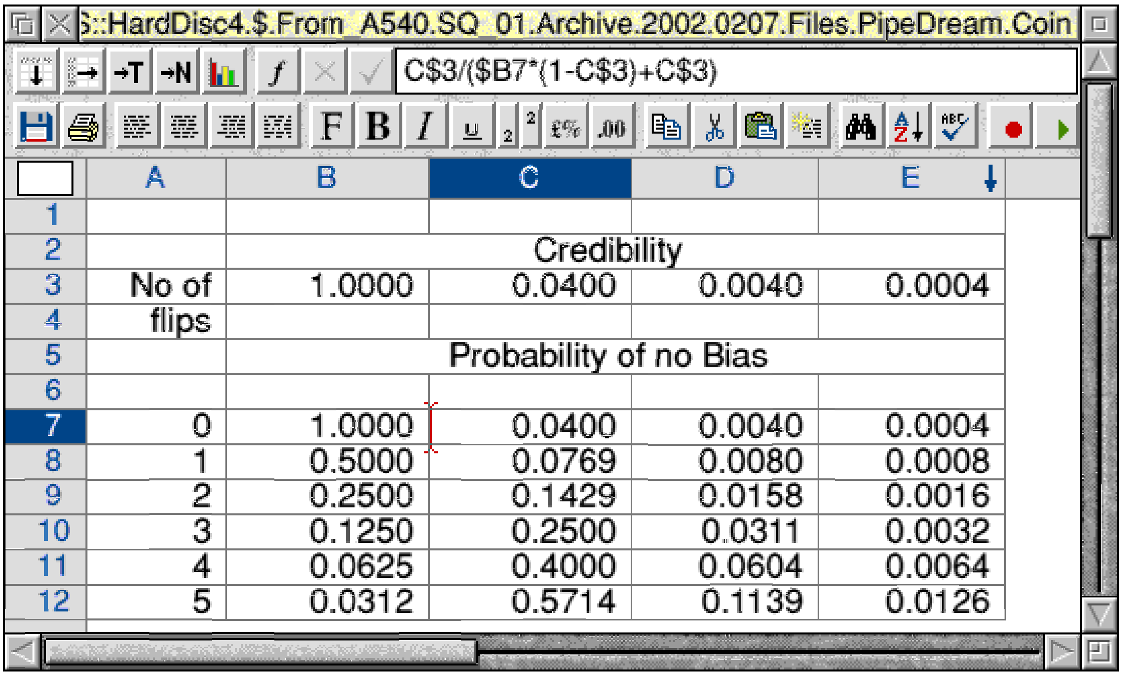
After the second head comes up we have lost forever another 24% leaving us with 24% which together with the constant 4% (28% in total) is for future allocation. We can’t go back and change what happened. Of our original 100%, 72% has gone for ever.

What has happened to the original 100% after the fifth head? We have ‘used up’ 93% of it. That 93% has fallen victim to indisputable knowledge gained by experiment. Of the remaining 7% only 3% supports our ‘unbiased coin’ PDF. The 4% which started life representing our ‘a priori’ scepticism is now 4/7ths (57%) of what is left.

Even now the coin could be unbiased; we might have had a run of bad luck. Nevertheless our ‘a posteriori’ situation is that the split is 4:3 in favour of scepticism. Phrasing this differently, if your initial level of scepticism was equivalent to the 4% figure, five heads will have convinced you that skulduggery is more likely than not. Your original level of scepticism, 4%, has been magnified by the experimental evidence to become 57%.

Is the coin unbiased?

Although I used PipeDream to create my credibility graphs it is relatively easy to set up any spreadsheet package such as Schema or Eureka to do the same thing. The screenshot below is of the PipeDream spreadsheet, [Coin]. The first column shows the number of times the coin comes up heads. The second column is the probability of getting this number of consecutive heads using the unbiased coin PDF. The final three columns are the interesting ones; I used the values in them to draw the three lines on the graph.



	A	B	C	D	E
1					
2		Credibility			
3	No of	1.0000	0.0400	0.0040	0.0004
4	flips				
5		Probability of no Bias			
6					
7	0	1.0000	0.0400	0.0040	0.0004
8	1	0.5000	0.0769	0.0080	0.0008
9	2	0.2500	0.1429	0.0158	0.0016
10	3	0.1250	0.2500	0.0311	0.0032
11	4	0.0625	0.4000	0.0604	0.0064
12	5	0.0312	0.5714	0.1139	0.0126

I had considered including versions of this spreadsheet in many formats such as Schema, Eureka, Fireworkz and Excel, however, on a previous occasion when I asked about this I received no positive response. Indeed those who decided to create a spreadsheet told me that they preferred to do their ‘own thing’. Is this the general consensus?

The formula in B8 is 0.5*B7. This formula is replicated down the column. The values are halved with each throw of the coin. For example, the probability of five heads in a row when using an unbiased coin is 1 in 2^5 or 3.12%. The entry in B12 is 0.0312.

The values in Column C are those used to generate the line corresponding to a credibility of 4%. From the screenshot you will see the formula which I have used. It’s not as bad as it first appears. The formula in C7 is replicated down and across the spreadsheet.

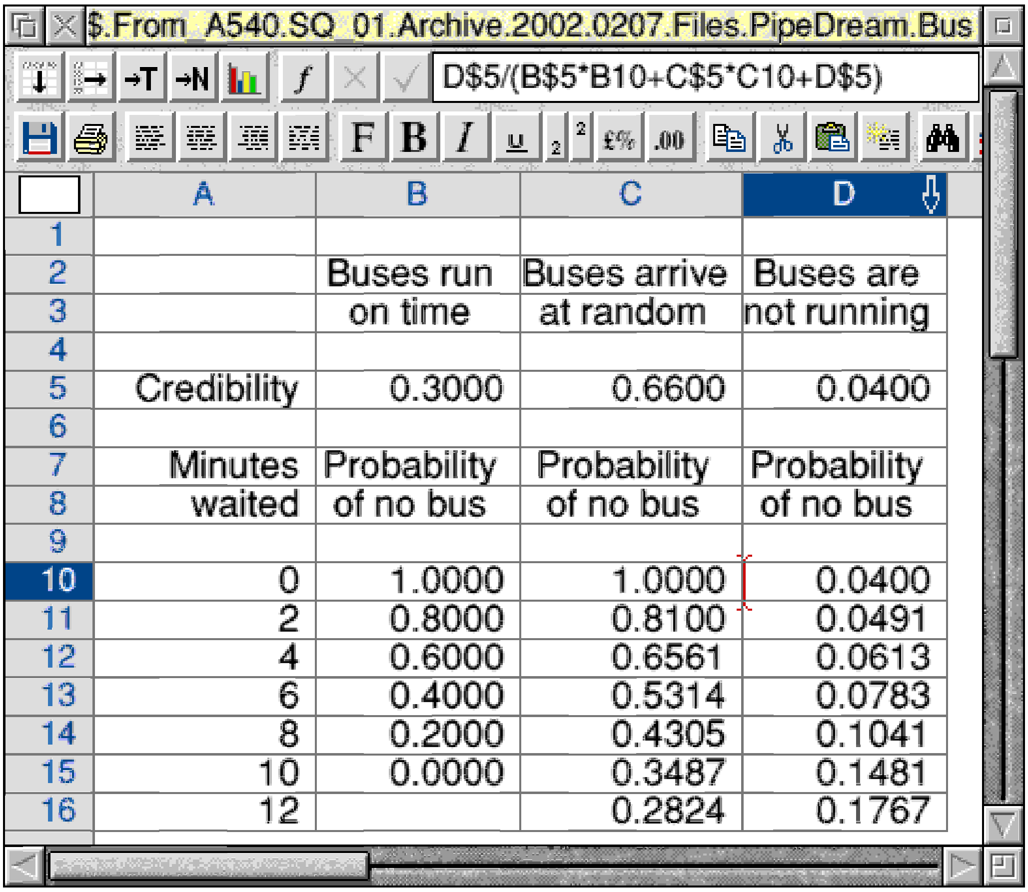
The value of (1 − C3) is the ‘a priori’ probability that the coin is unbiased. The value in B7 is the proportion of (1 − C3) which has not been used up. B7*(1 − C3) is the ‘a posteriori’ portion of the original probability for which the outcome could still be a tail at sometime in the future. The sceptical bit, C3, has not been used up, so the amount of probability still around for allocation is B7*(1 − C3) + C3. The ratio C3/(B7*(1 − C3) + C3) is the ratio of the values (coin is biased)/(don’t know yet). The value of this ratio can be regarded as the ‘a posteriori’ probability that the ‘coin is unbiased’ PDF does not apply.

Looking at the spreadsheet table you’ll see that after five heads this probability has increased from 4% in C7 to about 57% in C12. The original degree of scepticism, 4% in C7, has now been magnified to 57%; scepticism about the unbiased nature of the coin has overtaken belief.

If your initial level of scepticism was lower, say 0.4% rather than 4%, then it will take another three throws of the coin before your suspicions dominate what is still possible. It will take yet another three if your rating was 0.04%. I have to add that this value, 0.04%, is such a low figure that it is unlikely to be the result of any objective assessment.

Will the bus ever come?

I used the spreadsheet in the screenshot below to generate the probability graph for the bus.



	A	B	C	D
1				
2		Buses run	Buses arrive	Buses are
3		on time	at random	not running
4				
5	Credibility	0.3000	0.6600	0.0400
6				
7	Minutes	Probability	Probability	Probability
8	waited	of no bus	of no bus	of no bus
9				
10	0	1.0000	1.0000	0.0400
11	2	0.8000	0.8100	0.0491
12	4	0.6000	0.6561	0.0613
13	6	0.4000	0.5314	0.0783
14	8	0.2000	0.4305	0.1041
15	10	0.0000	0.3487	0.1481
16	12		0.2824	0.1767

I have constructed it in a slightly different way from the one for the coin but the principle is identical. The advantage of this new layout is (a) that you can apply it more easily to other scenarios and (b) it can be extended to any number of possible PDFs. You will remember that in this hypothetical I selected two possible PDFs. One was that the buses run on time, the other is the PDF for buses arriving at random with a mean interval of ten minutes. You can add many more PDFs, indeed, as many as you can conceive. I shall discuss the use of multiple PDFs further in my philosophical bit.

The values in columns B and C from row 10 downwards are the probabilities calculated for these two selected PDFs. The values in row 5 are the ‘a priori’ probabilities associated with each PDF. In the screenshot you can see the formula in D10. It is replicated down the column for as many rows as you need. You will see that the denominator is the sum of products. Do you remember the Addition and Multiplication Rules from last month?

The value in column D returned by this formula is the ‘a posteriori’ probability that there will never be a bus. It rises to 0.5 (50%) at about 26.6 minutes. Hence, after 26.6 minutes the calculated ‘a posteriori’ probability that a bus will come at sometime in the future is always less than the probability that it will never come. Have another look at the graph for the bus; it is column D plotted against column A. You will see what I mean.

The Philosophical Bit

In the jargon used by philosophers what I am about to describe is known by the grandiose title: “The Probabilistic Epistemology of Knowledge”.

I prefer to call it “Bayesian Epistemology” because the mathematical content is based on Bayes’ Theorem. This philosophical subject is about devising a rational basis for changing your belief about such things as buses, coins and even your faith in the scientific method. If you are really interested then a book called “Judgement under Uncertainty: Heuristics and Biases” published in 1982 by Cambridge University Press might excite a few of your more active brain cells.

Nothing about the future events of this world is known nor can be known for certain. We live in a world of probabilities. Since the seventeenth century a distinction has been made by philosophers between ‘a priori’ and ‘a posteriori’ knowledge. The term ‘a priori knowledge’ as used by philosophers is not what most of us would call knowledge; it is what we would generally refer to as belief without evidence. One example of a ‘great belief’ is the Axioms of Mathematics; another is belief in the Laws of Logic.

What I have called Bayesian Epistemology is a mathematical technique for modifying such ‘a priori’ degrees of belief in the light of evidence gained by experience. This method of modifying our degree of belief has the dubious merit of being recognised as rational by philosophers! I suspect that the anonymous French philosopher to whom I referred last month subscribed to this philosophy.

My contribution is this. With the possible exception of scientists toeing the party line, each of us have degrees of belief about different theories and we have them simultaneously. As a general rule Scientists test only one theory at a time. For the rest of us belief is rarely, if ever, total. In the case of the missing bus we can believe that buses run on time, that buses arrive at random and that a bus will never come. We can believe all these things at the same time; we can do so by having a different degree of belief about each potential PDF.

What is important to us is not which theory is ‘true’ but when, if ever, the bus arrives.

The amount of evidence needed to persuade you to change your degree of belief depends on how prejudiced you are (or how much faith you have) to begin with. This prejudice or faith can be quantified I have suggested. The greater your faith then the more contra evidence you will require before you change your mind. When it comes to missing buses or flippin’ coins the consequences of being wrong are relatively trivial. I am sure that you can all think of causes much nearer to your heart than my flippin’ coin so I shall not risk an argument by citing examples discussed on the Archive 42 list!

Finally

I am sure that those whom I have described as being of a ‘scientific persuasion’ and those who have been taught classical Hypothesis Testing will disagree totally with my inclusion of degrees of belief in many theories simultaneously in mathematical analysis. My defence is that I am equally sure that most people think in the way that I have described namely that we have initial ‘a priori’ degrees of belief in many conflicting theories simultaneously.

What is less controversial is that we are all more of less willing to modify these degrees of belief in the light of experience. This assertion does have a small measure of support from those philosophers who subscribe to the Bayesian Epistemology of Knowledge. However, as far as I know, this philosophy concentrates on only one belief at a time; discussion relates to the absolute truth or otherwise of this one belief.

If you wish to comment then please email or write to me or submit a note to Paul for inclusion in his Comment Column. I assure you that I shall not get upset by criticism. My ‘a priori degree of belief’ in what I have written is far too strong! Indeed I am surely prejudiced beyond redemption.