## Gerald's Column by Gerald Fitton

In this series of articles I hope to help you understand words such as 'probability' and discover how statistics can be applied to problems that have an uncertain outcome.

This month I shall describe the Addition and Multiplication Rules.

### The flippin' coin

The students who attended my course in Risk Assessment were studying for a senior management qualification. They were not teenagers; indeed some were older than me! They had all 'been around'. All had full time 'risk taking' jobs. The coin I flipped seven times was a pre 1920, sterling silver half crown which I had inherited from my grandfather. On seven consecutive throws it came down Heads.

If you chose as your mathematical model an unbiased coin then the probability of getting eight heads in a row is 1/256. I asked what the experimental result of seven heads in a row tells you about the probability of a head on the eighth throw? Let me rephrase my question. Would you bet on Tails for the eighth throw? Would you bet on a Tail if I offered you odds better than evens? Would you bet on Tails if I offered you odds of 10:1?

The Multiplication and Addition Rules are relevant to your decision.

### Recapitulation

Before I tell you why let me remind you about the missing bus. At first we guessed a PDF (Probability Distribution Function) based on the assumption that the 'buses run on time' (every ten minutes). This PDF failed after ten minutes because no bus came.

We developed another PDF based on randomised bus times with a mean spacing of ten minutes. Using this PDF and my trusty computer I calculated that there was a 96% chance that a bus would have arrived in 30 minutes—and still no bus arrived.

I suggested to you that instead of thinking about this 96% probability as the probability that the bus should have arrived you should shift to your focus to the probability that the 'random bus times' model is invalid. It is this subtle shift in what the probability means which is at the heart of many scientific crucial experiments.

### Did I mislead you?

I haven't been quite fair to you. I suggested to you that the 96% probability could be considered as the probability that our 'random buses' PDF was invalid. My defence is that I didn't want to confuse you by introducing these wonderful Rules at that stage. Also the only PDF I have mentioned for the flippin' coin is one based on a totally hypothetical unbiased coin which has exactly 50% probability of coming down Heads. With as few as five heads in a row you might quantify your doubts about the coin being unbiased as "This model has a 97% probability of being invalid"—but would accept that evaluation?

If your thinking is fairly representative then you will be more willing to think that we have the wrong PDF for the bus after half an hour than for the coin after five heads (even though the probability values are about the same—96% and 97% respectively). One of my aims for this article is to show you how you might rationalise and even quantify that prejudice.

### To Multiply or Add

The things which are either multiplied or added are probabilities. These two rules are limiting cases of another formula which usually goes under the name of Bayes' Theorem. In all the text books I've read this Theorem is usually explained so badly that even hardened statisticians fail to remember its syntax. I am convinced that many who might be regarded as experts don't understand it properly!

### **Statistical Independence**

The Multiplication Rule can be applied only when the outcomes of two separate and distinct events are totally and completely 'Statistically Independent'.

In the case of the missing bus, if the buses run exactly on time then the probability of the bus arriving in the second minute is not Statistically Independent of the non arrival of the bus in the first minute. Using the 'buses run on time' PDF the probability of a bus arriving in the second minute is increased by its earlier non arrival; so you can't use this Rule.

In the case of the flippin' coin you can apply the Multiplication Rule to calculate the probability of two heads in a row if and only if you believe that the result of the second throw is in no way influenced by the result of the first throw. Statistical Independence from throw to throw is a necessary condition for using the Binomial Distribution Function to calculate the separate probabilities of the six possible outcomes of a five throw event.

If the probability of a head in a single throw remains constant with a value p, then the probability of getting two heads in a row is p times p. The Multiplication Rule can be applied time and time again provided all the outcomes are Statistically Independent. The probability of getting eight heads in a row (which did happen) is (p^8).

Hint: If p = (1/2) then  $p^8 = (1/256)$ ; if p = 1 then  $p^8 = 1$ .

### **Mutually Exclusive**

At the other end of the Bayes' Theorem scale we find the Addition Rule. It can be used when the outcomes are Mutually Exclusive. It is often used when an event has many outcomes and we are willing to accept several of those outcomes.

An example is throwing a dice—by the way I do know that 'real mathematicians' call them 'die' in the singular but, to me, 'die' seems so contrived. Let us suppose that what I am looking for is an even number. There are three of them on a dice, 2, 4 and 6. Each of these three outcomes has a probability of (1/6). Using the Addition Rule the probability of an even number is (1/6) + (1/6) = (1/2).

The Addition Rule can be used because it is impossible to get 2, 4 and 6 on the same throw. All three of our desirable outcomes (2, 4 and 6) are Mutually Exclusive.

Contrast this with the flippin' coin. We can not add the 50% probability that we get a head on the first throw to the 50% that we get a head on the second throw because we could get a head on both throws.

#### **Bayes' Theorem**

I may treat you to my explanation of this Theorem another day. If you wish to help me with my future article then write to me about it—I shall be most interested. Today however I really do want to get to my rationalisation of the differing degrees of credibility (faith?) which we have in the assumed PDFs of the missing bus and the flippin' coin.

#### The Statistically Independent Buses

Our second PDF for the missing buses was calculated by assuming that the probability that a bus would not arrive in any one minute interval remained constant at 0.9. We calculated the probability that no bus would arrive in 30 minutes. We did so by repeated use of the Multiplication Rule formula  $(0.9^{30}) = 0.042$ . This about 4%. The complement, 96%, is the probability that at least one bus will arrive during those 30 minutes.

Using a simple spreadsheet (rather than a calculator) the probability of not getting a tail in eight throws of the flippin' coin  $(0.5^8) = 0.0039$ . This is about 0.4%, ten times smaller than the probability of no bus in half an hour. So why do you need eight heads, rather than five, before you decide to 'give up' the unbiased coin model?

### **A Priori Probability**

I want to introduce you to something called 'A Prori Probability'. My not-quite-accurate description of this probability is that it is the probability which you allocate to an outcome before you have thought about it! Let me use the missing bus as my first example.

Last month I followed the usual practice of choosing one PDF at a time. This is the method taught to scientists and others wishing to test a hypothesis. First I chose the 'buses run on time' PDF; then I moved on to a 'random bus times' model; finally I accepted my friend's model of 'no bus'.

I want to introduce you to something which doesn't appear in any of the statistical text books I have read but something which, many years ago when I was a student, I found in a book by a philosophical mathematician. Unfortunately I can't remember who; certainly it was somebody famous who had a statistical theorem named after them. The book was written in French; I recognised the author's name so I bought it from the secondhand book shop for a 'tanner'. His arguments, in French, were so clear that even I could follow them!

I know I'm going to come in for a lot of criticism for repeating (and agreeing with) his philosophy. I am expecting those who subscribe to what I might call a 'scientific persuasion' will be amongst my greatest critics because they don't do things this way.

## **A French Philosophy**

I'm about to leave home to walk to the bus stop. You will remember I have to get my book to the library today or face a fine. Have I time for a cup of tea or must I leave now? I invite you to join me (do you take your tea with or without sugar?) and, together, we'll ponder on the probability of a bus arriving within 30 minutes of my arrival at the bus stop. By the way, I have some deliciously scented Earl Grey tea!

The bus time table shows the buses arrive at 10 minute intervals. Shall we accept that as an approximate mathematical model? We certainly should consider it. On the other hand the buses might be randomised—perhaps we shouldn't rule that out? In return for your (hypothetical) Earl Grey (surely without milk and sugar) tell me, which would you choose?

You don't know. Neither do I. Like the unknown French philosopher whose book I read during my impressionable youth, I don't want to bet my book fine exclusively on either. I would like to take account of both possibilities. If together we can find a mathematical way of doing this then that will be more realistic; it will mirror the way people like you and I (but not scientists?) would approach the risk of not catching a bus in time to pay my fine.

### I didn't think of that

Following our anonymous philosopher we shall assign an a priori probability to the various models we can think of. We'll allocate an 'a priori' probability of 30% to the 'buses run on time' model and 50% to the 'random bus times' model. These two models are mutually exclusive so I can use the Addition Rule and add these probabilities together. I'm sure that you don't need a spreadsheet to discover that 30% + 50% = 80%.

If you are a curious person—and you must be or you wouldn't be reading this—you will be wondering what happened to the other 20%. My variation on this a priori philosophy is to allocate the other 20% to an "I didn't think of that" model of the bus arrival times!

I do want to emphasise that (scientists apart?) many people think this way even if they don't name and quantify these a priori probabilities. Nevertheless I shall expect to hear 'bad things' from many scientists (yes please) who disagree with me! If you have interesting but extensive things to say then perhaps the Comment Column might be a good place for you to be as critical of me as you wish—and as I deserve.

For all those curious people who have not given up in disgust at my inclusion of the mysterious French philosopher's a priori scenario and my personal contribution (the "I didn't think of that" variation) I shall continue.

## **Rules Rule OK?**

The probability of the bus arriving in 30 minutes can be calculated using a mixture of the Addition Rule (as above) and the Multiplication Rule (as I shall describe below).

If the 'buses run on time' PDF is true then it is 100% certain that a bus will arrive within 10 minutes—and hence certain that a bus will arrive in the first half hour.

We multiply the 30% a priori probability for this PDF by this 100% and we get 30%.

If the 'random bus times' model is true then (as we discovered last month) there is a 96% probability that a bus will arrive within 30 minutes. We multiply the 50% a priori probability for this PDF by the 96% and we get 48%.

If the "I didn't think of that" scenario is true, well, anything could happen. To be on the safe side (because I don't want to pay the fine) I shall assume that the outcome from this scenario is complete and utter disaster! I shall assume that the bus doesn't arrive within the 30 minute period. I multiply the 20% a priori probability by 0%. Once again you won't need a spreadsheet to discover that the answer is zero.

Finally the Addition Rule: 30% + 48% + 0% = 78%.

As we sit at home supping our cups of delicious Earl Grey (have a chocolate biscuit?) we conclude that the probability of a bus arriving within the first 30 minutes is 78%.

### **Spreadsheets**

If I don't say something about spreadsheets in this article then I shall get a reprimand from Paul—and he might cancel the rest of this series. So here goes.

A spreadsheet package (I used PipeDream) is a wonderfully simple way of generating the graph shown below. Nowadays I wouldn't try to do it with only a calculator. This month I shall limit myself to interpreting it for you. Next month (DV) I shall explain how to do it.



#### It's Incredible

The compound event is flipping the flippin' coin up to 20 times. It comes up heads every time. You will see that the y-axis is labelled "Probability of Bias" and that the graph contains three S shaped curves. The parameter which varies from curve to curve is what I have called "Credibility"—but perhaps I should have called in "Incredibility"!

The left most curve has this parameter set to 4%, the second and third curves have values 10 and 100 times less. What is the meaning of this parameter? It is the quantitative probability which I have allocated to the possible "I didn't think of that" scenario.

Think of it this way. The a priori probability of the coin being (nearly) unbiased must be high. A value of 96% for this a priori probability corresponds to a 'credibility' of 4%. If your personal choice is 96% it implies that you believe the a priori probability of some form of deception is 4%. If the value you choose is as high as 99.96% then this credibility parameter is as low as 0.04%.

You might say that the value you choose is your personal estimate of my 'credibility'.

### When to change your mind

For the three lines on the graph you will see that the "Probability of Bias" passes through the value 0.5 when the number of heads in a row is about 5, 8 or 11. The value 0.5 (50%) is the point at which the probability of deception exceeds the probability of a fair coin.

### The Philosophical Bit

However much faith you place in your choice of PDF (and scientists tend to stick to one PDF at a time) there will come a time when the evidence against your pet theory mounts up to a level that even you can not accept. With a high value of a priori probability it will take a lot of evidence to convince you that you should you 'give up' and revert to 'Plan B'.

"One man's faith is another man's prejudice".

### The probability of rare events

My lecture on this subject had the title "Pigs might fly". Let's save most of it for another day. Right now I can not restrain myself from asking just one short question.

Some of you might be asked to write reassuring remarks about the probability that a nuclear reactor will melt down and suggest that it might happen only once in a billion years. You might be asked to do probability sums on all sorts of unwelcome rare events ranging from contamination by GM crops to a nationwide failure of the national grid.

My question to you is this: Have you included an a priori probability for the "I didn't think of that" scenario? In the case of rare events the possibilities you haven't thought of assume an overwhelming importance and tend to swamp the chains of events you can contemplate.

# Finally

I know that those who I have described as being of a 'scientific persuasion' and those who have been taught classical Hypothesis Testing will probably disagree with the French philosophy which impressed me so long ago. If so then please email or write to me or submit a note to Paul for inclusion in his Comment Column. If you have any comment you think might interest me then please let me know.