# *Gerald's Column*
## *by Gerald Fitton*

Last month I started answering a Mathematics problem set by Andy Marks. Andy's Problem has sixteen different solutions. Last month, I discussed a similar but simpler problem which has only one solution. This month I shall describe how last month's method can be extended to problems, like Andy's, which have multiple solutions.

### The Largest Prime Number

A couple of months ago I set a little puzzle. I said:

"I have always enjoyed Colin Singleton's Puzzle Corner so let me set one of my own. What is the largest prime number which can be represented by the FPE? This number represents a 'sort of' upper limit for those simple encryption methods in which numbers are represented in FPE format(s) rather than long character strings."

I have received one reply—from Colin! I won't tell you his answer (yet) but it consists of sixteen digits starting 9,007, and ending 881.

Colin says:

"The IEEE Double Precision Standard holds the mantissa of a floating point number in 53 bits (of which 52 are 'real' and 1 is implied) so it can hold numbers less than 2^53 with absolute precision. This standard is used by BASIC 64, SciCalc and all Spreadsheets."

Colin has an 'old' machine with an 'old' version of the FPE. I know that when I did spreadsheet sums a few years ago on my 'old' A440 and A540 I did get that kind of precision. However, nowadays, with more recent versions of the FPE I don't always get a precision equivalent to the IEEE Double Precision Standard. Have a look at my article in the April 2001 issue of Archive for more details.

Please let me know if you have a result which is much smaller than Colin's. Also let me know if you arrive at a number which you think might be the same as Colin's and I shall reply by telling you whether it is or not.

Eventually I shall publish Colin's number. By the way, what's happened to Puzzle Corner?

### Andy's Problem

Back to the problem which Andy posted on Archive-on-Line. First a reminder:

"A theatre has a show running where tickets are £9.50 per adult and £5.75 per child. On one night, the total takings was £3,625. How many children's and how many adult's tickets were sold that night?

What Andy wanted was what I might call an Analytical Solution rather than a Numerical Solution, that is a Trial and Error solution.

## Natural Numbers

One way of expressing Andy's Problem Mathematically is to say that he requires the solution (the values of x and y) which satisfy the equation: a*x + b* y = c. The difficult feature of this problem for those who have been taught Mathematics in a more recent school than that at Alexandria (circa 250 AD) is that x and y are Natural Numbers.

For those of you who are interested, in 1998 I wrote a set of articles for Archive about different sorts of number. Five of those types of number are shown in the table below.

| | | | | | |
|---|---|---|---|---|---|
| $\mathbb{N}$ | = | $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$ | | = | Natural (counting) numbers |
| $\mathbb{Z}$ | = | $\mathbb{N}$ | $+ \{0, -1, -2, -3, \ldots\}$ | = | Integers (whole numbers) |
| $\mathbb{Q}$ | = | $\mathbb{Z}$ | $+ \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots\}$ | = | Rational numbers (fractions) |
| $\mathbb{R}$ | = | $\mathbb{Q}$ | $+ \{\pm\sqrt{2}, \pi, e, \ldots\}$ | = | Real numbers (smooth continuum) |
| $\mathbb{C}$ | = | $\mathbb{R}$ | $+ \{\sqrt{-1}, (1 + \sqrt{-1}), \ldots\}$ | = | Complex (and imaginary) numbers |

Mathematicians refer to 'Domains'. Andy's problem and its solutions lie in the Domain of Natural Numbers whereas the Mathematical solutions to many problems about the Nature of the Physical Universe lie in the Domain of Complex Numbers. These days at school the tendency is concentrate almost exclusively on the Domain of Real Numbers.

One consequence of restricting the domain to Natural Numbers is that fractions (look at the table) do not exist and so an operation of Division which returns fractions does not exist.

What we can do in the Domain of Natural Numbers is express our answer to say, 14÷3, as two Natural Numbers, a Dividend and a Remainder. The Dividend is the Natural Number 4 and the Remainder is 2. Modulus Mathematics is about the properties of Remainders.

As a 'by-the-way' you will see that zero is included as one of the Natural Numbers—it wasn't always the case. The difficulty with including zero can be expressed as a question: "How do you count the number of people in a room when nobody is there?"

## Andy's Problem

Andy's Problem can be stated Mathematically as:

"Solve 950*x + 575* y = 362500 where x and y are Natural Numbers."

I have converted all the money (Prices and Takings) into pennies. I wonder if someone looking at this twenty years from now will wonder what Sterling was all about! Perhaps we should have relocated Andy's Problem on the Continent of Europe and do it all in Euros. I have a problem with that. I'm not sure what a hundredth of a Euro is called.

The spreadsheet which I used last month called [Unique] works only if the problem has a unique solution. For this reason it can't be used to solve the unexpurgated version of Andy's Problem. In order to solve Andy's Problem we need to add a couple of stages 'up front' before we apply the Inverse Multiplier technique which I described last month.

## The Monthly Disc

The Archive monthly disc contains files in both PipeDream and Fireworkz format. You can enter your own numbers into these spreadsheets. If a solution exists then the spreadsheet will find it (if there is only one) or them (if there are multiple solutions). If a solution does not exist then the spreadsheet will return a suitable error message.

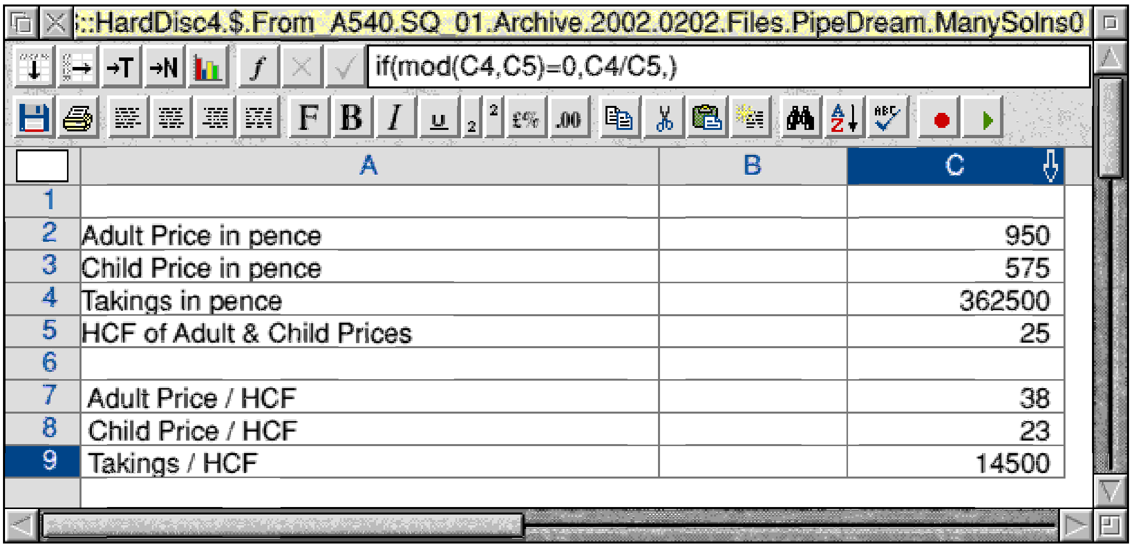
Fireworkz files will load into the demo version of Resultz which is available on our website or, if you are not 'connected', then we can send you a copy by post. If you have the Archive monthly disc with the spreadsheet files then double click on the file [ManySolns] and the supporting custom functions will load.

## Highest Common Factor (HCF)

One Mathematical difficulty with Andy's Problem arises because both the Adult Price (950 pence) and Child Price (575 pence) are divisible by 25. They are both divisible by 5 as well but that is not important. The important feature for you to appreciate about 25 is that there is no number larger than 25 which divides exactly into both 950 and 575.

When I was at school such a number was called the Highest Common Factor (HCF) of the two numbers. I have been told (by some of my correspondents) that in many more modern Mathematical books the HCF is referred to as the Greatest Common Factor (GCF)—but I was taught that it is, or was, called the HCF—so I shall stick with what I know!

I shall risk repeating that the method I have used to find the HCF is known as Euclid's Algorithm (discovered by him about 2300 years ago). It is that Algorithm which is built into my HCF custom function and called in cell C5 of the [ManySolns] spreadsheet.




| | A | B | C |
|---|---|---|---|
| 1 | | | |
| 2 | Adult Price in pence | | 950 |
| 3 | Child Price in pence | | 575 |
| 4 | Takings in pence | | 362500 |
| 5 | HCF of Adult & Child Prices | | 25 |
| 6 | | | |
| 7 | Adult Price / HCF | | 38 |
| 8 | Child Price / HCF | | 23 |
| 9 | Takings / HCF | | 14500 |

## Divide by the HCF

Have a look at the screenshot above. The HCF of the two prices is returned by my custom function (based on Euclid's Algorithm) in C5. Have a think about my next sentence. If there is to be any solution at all then the Takings must be divisible exactly by this HCF.

In the block C7C9 I divide the Prices and Takings by this HCF.  The results of the division are 38, 23 and 14500.  I assure you that the new equation, 38*x + 23* y = 14500,  has exactly the same sixteen solutions as the original equation, 950*x + 575* y = 362500.

## The Multiplicative Inverse

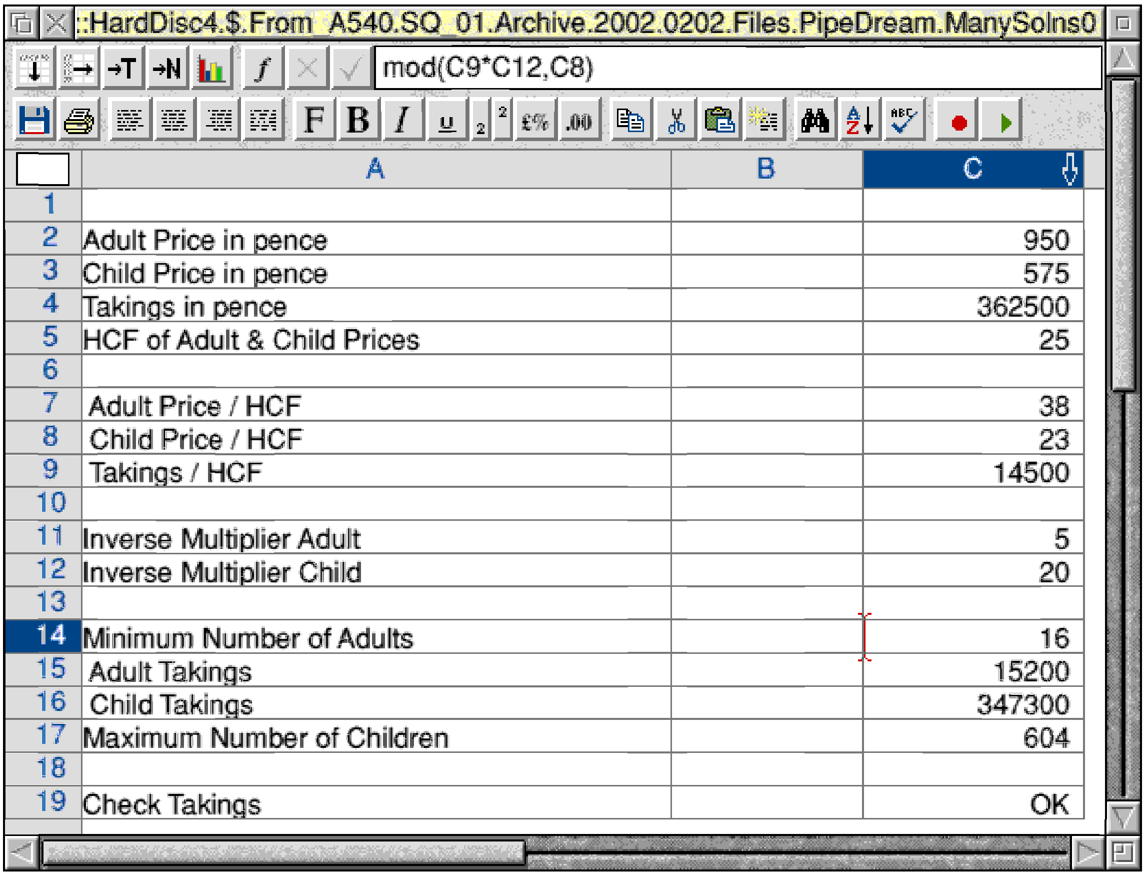
We are ‘better off’ with  38*x + 23* y = 14500  than with 950*x + 575* y = 362500.  We are better off because we know that the coefficients, 38 and 23, are relatively prime.

By dividing the prices in C2 and C3 by the HCF we have ensured that the coefficients in C7 and C8 are relatively prime (their HCF is 1).  Have you remembered that a unique Multiplicative Inverse exists if and only if C7 and C8 are relatively prime?

Have a look back at last month’s article you will recognise that the numbers 38 and 23 are those which I chose for my example and you may remember (if not you can look up) that the Multiplicative Inverse of 38 (using modulus 23) is 5 and that the Multiplicative Inverse of 23 (using modulus 38) is 20.

## The Minimum Number of Adults

The number of Adults is calculated in exactly the same way that we used last month.  You can see the formula in the formula line of the spreadsheet below and you will see that the answer returned is 16 (just like last month).  However, there is a difference this time.  This month’s answer, 16, is the Minimum number of Adults and not the Unique answer.




| | A | B | C |
|---|---|---|---|
| 1 | | | |
| 2 | Adult Price in pence | | 950 |
| 3 | Child Price in pence | | 575 |
| 4 | Takings in pence | | 362500 |
| 5 | HCF of Adult & Child Prices | | 25 |
| 6 | | | |
| 7 | Adult Price / HCF | | 38 |
| 8 | Child Price / HCF | | 23 |
| 9 | Takings / HCF | | 14500 |
| 10 | | | |
| 11 | Inverse Multiplier Adult | | 5 |
| 12 | Inverse Multiplier Child | | 20 |
| 13 | | | |
| 14 | Minimum Number of Adults | | 16 |
| 15 | Adult Takings | | 15200 |
| 16 | Child Takings | | 347300 |
| 17 | Maximum Number of Children | | 604 |
| 18 | | | |
| 19 | Check Takings | | OK |

Look at cells C7 and C8. Can you see that the Takings for 23 Adults (at 38 pence each) will be exactly the same as the Takings for 38 Children (at 23 pence each)? In both cases the contribution to the Takings will be 23 * 38 = 374 pence. Every time we add another 374 pence to the Takings we can 'add in' another 23 Adults or another 38 Children.

### The Minimum Number of Children

I won't bother with another screenshot. The minimum number of Children is 34, the same as the answer (to the simpler question) which we found last month.

### Multiple Solutions

The 'basic' solution is the one which we calculated last month. It is 16 Adults at 38 pence each plus 34 Children at 23 pence each. This totals 1390 pence. Every time we add another 374 pence to the 'basic' Takings of 1390 we can 'add in' either 23 more Adults or we can 'add in' 38 more Children. This does not mean that the only solutions to such a problem is when the Takings are of the form 1390 + n*374. There will be solutions when the takings are t + n*374 but the 'basic' solution will not be 16 Adults and 34 Children but something else. It is quite instructive to see what a large proportion of numbers close to 1390 (for the 'basic' Takings) all give completely valid 'basic' solutions.

You will remember that we divided through the original equation by the HCF (25) of the Prices. The statement in the previous paragraph can be 'translated' to the higher prices as: "Every time we add 374*25 pence to the Takings we can 'add in' either 23 more Adults or 38 Children". The 'basic solution' is still a minimum of 16 Adults and a minimum of 34 Children but at the original prices of 950 pence and 575 pence each. The Takings are not 1390 pence but 1390*25 pence.

### The Solutions

The number of Adults can be any number taken from the series: 16, 16+23, 16+(2*23), etc. The number of Children account for the remainder of the Takings. The number of Children will be a number from the series 34, 34+38, 34+(2*38), etc.

There are 16 different solutions. The general solution can be expressed as:
Number of Adults is (16 + 23m)
Number of Children is (34 + 38n)
Where (n + m) = 15

### Acknowledgements

I have to thank Rex Palmer and Tonnie Demarteau for their contributions to the layout and content of the [ManySolns] spreadsheet. A particularly useful contribution is the way in which the error messages are displayed in column B rather than 'messing up' column C.

My thanks also to Steve Ellacott of the University of Brighton for his help in providing references and inspiration as well as checking my Mathematics.

## Conclusion

This month we have solved Andy's Problem, 950*x + 575* y = 362500.  It has sixteen different solutions one of which has x =16 and another has y = 34.

Andy's Problem is a Problem in the Domain of Natural Numbers.  It does not use a Trial and Error technique but an Analytical technique.  This Analytical technique requires us to calculate the 'Multiplicative Inverse' of the coefficients in the equation.  Finding the Multiplicative Inverse in Modulus Mathematics is the Natural Number equivalent of Division in the Domain of Real Numbers.

Someday, if you 'talk to me nicely', I might tell you how to 'divide' in the Domain of Complex Numbers—that too requires you to multiply rather than divide (you multiply by what is called the Complex Conjugate).  Also some of you may know about the Mathematical entity called matrices (forces within solid bodies are best expressed as a matrix).  Division does not exist in Matrix Mathematics—the equivalent is to multiply by what is called the Inverse Matrix.

So you will see that in the wider world of Mathematics in which the operation of Division is often 'outlawed' there is a way around the difficulties which this deficiency causes.  The solution is to use multiplication just as we have in this case where our Domain is the Domain of Natural Numbers.

## Finally

You can email or write to me at the addresses given in Paul's Fact File.

Please note that we no longer have a telephone or fax.