Gerald's Column
by Gerald Fitton

Over the last few months I have spent some time explaining to you the shortcomings which are intrinsic to the nature of all digital computers when it comes to handling numbers. In a somewhat provocative way I have suggested that that the only type of number stored in a digital computer is the Natural number from the set $\mathbb{N}$. Of course you can cheat and, as I described in an earlier article, you can use modular arithmetic to store negative Integers from the set $\mathbb{Z}$. There are a few Rational numbers from the set $\mathbb{Q}$ which can be represented exactly in floating point notation but the rest are stored approximately. Real numbers, $\mathbb{R}$, fill all the gaps on the Number line between the Rational numbers. According to Colin Singleton a mere 18 437 736 874 454 810 623 Rational numbers can be stored in any digital computer which limits itself to the IEEE Double Precision Format – and a mere 18 437 736 874 454 810 623 numbers leaves a lot (an infinite lot) of gaps!

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, \dots \} \qquad = \text{Natural (counting) numbers}$$
$$\mathbb{Z} = \mathbb{N} \quad + \{0, -1, -2, -3, \dots \} \quad = \text{Integers (whole numbers)}$$
$$\mathbb{Q} = \mathbb{Z} \quad + \{¼, ½, ¾, \dots \} \qquad = \text{Rational numbers (fractions)}$$
$$\mathbb{R} = \mathbb{Q} \quad + \{\pm\sqrt{2}, \pi, e, \dots \} \qquad = \text{Real numbers (smooth continuum)}$$
$$\mathbb{C} = \mathbb{R} \quad + \{\sqrt{-1}, (1 + \sqrt{-1}), \dots \} = \text{Complex (and imaginary) numbers}$$

The inability to store these missing 'in-between' numbers is a serious shortcoming of the digital computer – and there is another. It is impossible to build a digital computer which can store all of the Natural numbers, $\mathbb{N}$ – there are too many of them!

This month, rather than pursue the elusive set $\mathbb{C}$ exclusively (as was my original intention) I hope to seduce you into accompanying me as I follow the trail signposted by these shortcomings. As usual, we shall allow ourselves to be sidetracked into an interesting digression or two along the way starting with some correspondence.

Infinity

Last month I said that "The set $\mathbb{R}$, the Real numbers, contains many more numbers than there are in the set $\mathbb{Q}$". I went on to say that "The number of Rational numbers is the ∞ sort of infinity whereas the number of Real numbers is of the $\aleph$ sort! These two sorts of infinity are sometimes called 'denumerable' and 'non-denumerable' because the number of numbers in the set can either be counted or not counted using the Natural numbers". I concluded this section with "It is the largeness of the $\mathbb{R}$ set which gives it its smoothness".

It is an intrinsic deficiency of all Digital computers that they are unable to store any representation of 'smoothness'. I shall return to this point later in my article.

I have received more than a few letters on the subject of infinity. The authors of these letters call me to task for not crediting them with a greater level of mathematical sophistication. My only excuse is that I was trying to simplify the subject – but it looks as though I am guilty of oversimplification. I have made that mistake before and, no doubt, I shall do so again before I put down my pen (or keyboard) for the last time!

Although I have slightly edited an email sent to me by Peter Chambers I believe I have retained the meaning. In this slightly edited form it reads:

"Cantor's $\aleph$ numbers usually are subscripted. So $\aleph_0$ is the countable infinity which most people assume is 'the' infinity. $\aleph_1$ is the 'next' infinity which is substantially greater than $\aleph_0$ (it may be generated for example by $2^{\wedge}\aleph_0$) the distinction being that one cannot map a set of cardinality $\aleph_0$ to a set of cardinality $\aleph_1$. Mathematicians do not know if c (the Continuum) maps to $\aleph_1$. It might but then again it might not.

"The ∞ symbol is not usually used in formal discussions of infinity due to the baggage it carries from pre-Cantor days.

"Interested readers should consult 'Infinity and the Mind' by Rudy Rucker for an accessible introduction and even more infinities."

Cardinality

Peter uses the term Cardinality. I think I have explained the meaning of Cardinality to you before but a little repetition won't hurt.

You can not do sums with Ordinal numbers such as 'Second' and 'Fifth'. The 'Second' and 'Fifth' people in a line when merged together do not have anything to do with seven.

You can do sums with Cardinal numbers. If you have five people in a room then the Cardinality of that set (of people) is the Cardinal number five. If one set of people has Cardinality two and another set of people have Cardinality five then, when merged, the enlarged set has Cardinality seven! I believe I have mentioned before that it was Bertrand Russell who used such an example to 'prove' that one and one make two!

I have said before (but not in Archive) that non mathematicians use Cardinal numbers as adjectives whereas mathematicians use them as nouns. In the sentence "Two people and five people together make seven people" the numbers are adjectives describing the number of people. You too can become a mathematician in two easy stages. The first is "Two of anything and five more of the same thing make seven of those things" – the numbers are still adjectives. Finally, "Two and five make seven" – now the numbers are nouns – now you are a mathematician!

Cantor studied the Cardinality of Infinite sets, not of people but of sets of numbers, points on a line and other infinite things. The Cardinality of a set is a noun.

For Finite sets you can find the Cardinality of the set by counting. Every one of you will be so familiar with the process of counting that you will have forgotten its wondrous nature. Let us carry out the 'thought experiment' of determining the Cardinality of a Finite set of people. We want to know how many there are. All you need to do is to hand out Ordinal numbers (imagine them written on pieces of paper) to everyone. Everybody gets one – nobody gets two. If you started with the Ordinal number 'one' then the last Ordinal number you hand out is the Cardinality of the set! Addresses and 'Offsets' in the memory of a computer are not Cardinal numbers but Ordinal numbers. The size of the memory is a Cardinal number. My A540's ROM has a Cardinality of 8 Mb!

Once upon a time I asked my son, David, if he thought I was a 'Computer person'. He placed three pennies on the table and asked me to count them out loud. I said, "One, two, three". He then proclaimed that this showed conclusively that I was definitely not a 'Computer person'. I asked him how the test worked. He took out the Snakes and Ladders game and said, "We'll store the pennies from square 25 onwards. Let's count the 'offset' from the (Ordinal) 25th square. Zero offset, one offset, two offset". His face took on a winsome smile I shall never forget, "Computer people start counting with the Ordinal number zero!". I wonder where he got that from – I suspect that he thought it up himself? Or did I once tell him that when he much younger? Certainly they don't teach it in school.

Infinite Sets

The subject Infinite series forms part of the A Level syllabus. It formed part of the Higher School Certificate – I took the last one in, I think, 1950 before A Levels were introduced. From these beginnings with Infinite series it took me about thirty years to get to grips with Infinity and understand its main properties.

I am pleased that some of my students learnt more quickly because of the difficulties I had. Let's see what I can do for you in a couple of pages of Archive.

One-to-One Correspondence

I suppose the single most difficult problem people have with infinity is knowing how big it is. This is because they try to use some variant of the 'Counting using Ordinal numbers' I described earlier as their sole method of getting to grips with it. There is another method; this alternative method relies on the concept of 'one-to-one' correspondence. Once you have grasped how it works then the nature of Infinity is within your grasp.

When you count using Ordinal numbers what you are doing is giving out one and only one Ordinal number to each element of the set. One-to-one correspondence means that each element in the set has just one Ordinal number ascribed to it and each of the handed out Ordinal numbers has been allocated to one and only one element of the set.

Mapping

Finding such a one-to-one correspondence between one set and another set is called Mapping. Finite sets (such as people in a room) can be mapped to a sub set of the Natural numbers. Some infinite sets can be Mapped to the infinite set of Natural numbers, $\mathbb{N}$ – but some, such as the Real numbers, $\mathbb{R}$, can not because there are too many of them.

So that you can see how Mapping works we'll have a digression.

Let us look at a function for which the correspondence is not one-to-one but two-to-one. The function $y = x^2$ maps values of x to values of y. The set of all possible values of x is called the Domain and the set of all possible values of y is called the Codomain. For the sake of this discussion let's say that the Domain and Codomain each contain nothing more than the set, $\mathbb{R}$, of all Real numbers. What you have to do is to consider every possible value of x and see what parts of the Codomain can be reached by the this squaring

function. It won't be long before you realise that only the positive half of the Codomain can be reached and that most of the time there are two values of x which map to each attainable value of y. The exception is zero which maps to only one value, zero.

Now let's have a look at the Inverse of this $y = x^2$ mapping. If you start with any positive number in the Codomain (a value of y) then you can certainly find a value of x which maps to that value of y – indeed, with only one exception, zero, you'll find two values of x (a positive and negative square root) from which you could have started. This 'square root' mapping, where it exists, is one-to-two.

Before you read this next bit you will need to open the refrigerator and take out a can of Heineken. Using the function $y = x^2$ there is no mapping from $\mathbb{R}$ in the Domain to negative values of $\mathbb{R}$ in the Codomain. Heineken has this remarkable property (or so I've been told) that it enables you to reach parts (of anything) which, without it, are unreachable. So, imbibe the Heineken read my last month's article and you'll realise that all you need to do is to extend the Domain to include numbers such as i. From what I shall call the $\mathbb{H}$ (for Heineken) extension of the Domain ($\mathbb{H}$ extends a Domain or Codomain from $\mathbb{R}$ to $\mathbb{C}$) you can reach negative parts of the Codomain that are unreachable from $\mathbb{R}$.

At this point (and after only one application of $\mathbb{H}$) you decide that you will make the Domain not $\mathbb{R}$ but the whole of $\mathbb{C}$. If the subtle change of variables from x and y to w and z doesn't throw you then look at mappings of the function $w = z^2$ from those parts of the Domain which I have called the $\mathbb{H}$ extension (the parts of $\mathbb{C}$ which are not within $\mathbb{R}$). What you find is that nearly all of this $\mathbb{H}$ extension has nowhere to map to in the $\mathbb{R}$ Codomain. Now you need another trip to the 'fridge; this second visit will inspire you to add the $\mathbb{H}$ extension to the Codomain so that the extended Codomain covers all of $\mathbb{C}$.

Now you have had two applications of $\mathbb{H}$. Do you need any more trips to the 'fridge? No! By extending the Domain and Codomain to $\mathbb{C}$, the Complex numbers, you have done something quite wondrous. Let me explain the nature of this wonder. For the function we have used, $w = z^2$, every point in the Domain can be mapped to a point in the Codomain; no surprise there. However, when we come to 'do it backwards' (invert the function to find square roots), every point (bar zero) in the Codomain can be mapped to two points in the Domain. Every point in the Codomain can be reached without any more applications of $\mathbb{H}$ and, even more remarkably, indeed somewhat wondrously, there is no point in the Codomain without at least one 'partner' in the Codomain.

The inverse function exists for all $\mathbb{C}$.

Now follows an important point about Complex numbers. There are many more functions other than $w = z^2$ but every single one, and many more that haven't been thought of yet, all have this wondrous property that the mapping from Domain to Codomain and back again will always be complete. No point in either Domain or Codomain will live in isolation – they will all have at least one partner. Complex numbers eliminate the need for more of the magic $\mathbb{H}$ because you can reach all the parts of both the Domain and Codomain that you could not reach when using $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. It is this property of Complex numbers which causes mathematicians to eulogise about them.

Cardinality

Back to the Cardinality of Infinite sets. The digression above was intended to make you more familiar with the concept of Mapping. We shall need that concept in what follows.

Nearly all the problems people have with the size (Cardinality) of infinite 'thingies' is because they try to count how big they are using the 'Ordinal number' method I described earlier. Don't worry about it. Don't even think about counting these infinite 'thingies'. Such efforts are doomed to failure, lead to confused thinking and will cause you to ascribe to the Infinite properties which it does not have.

Let me describe an alternative strategy which will help you to get a better feel for the features of sets having infinite Cardinality. The alternative strategy is to find something else for which you know the Cardinality; then see whether you can map your 'thingy' to the other thing of known Cardinality. If your 'thingy' has a one-to-one correspondence (Mapping) with the other thing then the Cardinality of your 'thingy' is the same as the known Cardinality of the other thing.

The starting point for measuring the Cardinality of infinite things is the Cardinality of the Natural numbers, $\mathbb{N}$. Peter Chambers (see above) confirms that Cantor used not the 'old fashioned' ∞ (which I rather like) but $\aleph_0$ for this Cardinality.

There is a fairly simple mathematical proof that the set of Real numbers, $\mathbb{R}$, will not map to $\aleph_0$. For this you need a larger Cardinality. Peter is in good company in that he doesn't know which $\aleph$ Cardinality you do need! He says that nobody knows! All is not lost. We do know that all the real numbers, $\mathbb{R}$, will map, with one-to-one correspondence to the Number line. The Cardinality of the set $\mathbb{R}$ is the same as that of the Number line.

Smoothness

Mathematicians know a lot about the Number line. Its Cardinality has been named as 'The Cardinality of the Continuum' and given the letter C (Peter says it should be a lower case c but I shall stick with my upper case C). I expect that at some time in the future C will go the same way as ∞. I shall mourn its passing.

If the set of Real numbers, $\mathbb{R}$, had a smaller (but still infinite) Cardinality than C then moving from one point to another would be jerky. If the Cardinality was the 'old fashioned' ∞ (which I still like) then moving from one point to another would be jerky. There is a mathematical proof of this which I won't bother you with even though I know I shall get somebody like Peter telling me how easy it would be to explain it to the sophisticated readership of Archive; however, I must move on or be edited out by Paul.

By their very nature digital computers are jerky and can never be programmed to handle those things about functions which depend on the smoothness of the transition from one point to the next. Right now you may not be able to think of anything that depends on that smoothness so I'll give you an example, but not yet.

Containing Infinity – 1

Before I get to my example I have to persuade you to accept another rather surprising fact. It is this. If you take any small part of the Number line, let's say the bit from –1 to +1, then you can set up a one-to-one correspondence between every point on the full Number line (the one which goes from –∞ to +∞) and your short bit of line.

There are many mappings which do this. Here is just one of them. Consider the function $y = arctan(x)$ where y is a Real angle between –90° and +90°. The values of x vary from –∞ to +∞. This $y = arctan(x)$ function has successfully mapped every Real number in the Domain between –∞ and +∞ to numbers between –90° and +90° in the Codomain.

You may think I'm cheating but I assure you that I'm not. When I worked at Vickers, following a simple instruction manual on the subject, I built and used a computer which contained this very mapping. I needed to be able to map the infinite to something I could get my hands on (well get a voltmeter on it to be more precise).

The computer I designed and built was not unique. At that time (the early 1960s) lots of people were building such computers, many of which were much more powerful than mine. I remember going down to Warmwell (a Nuclear Research Station) to program a highly sophisticated computer which contained many mappings of the infinite to finite voltages. All these computers differed from the digital computers with which we are so familiar today in one major respect. They were able to handle smooth changes in Real variables with no difficulty whatsoever.

Containing Infinity – 2

If I have been successful in persuading you that an infinitely large but smooth set of points having Cardinality C can be contained within a short piece of string, then let me try something a little more difficult.

You can carry out an infinite number of steps of a process within a finite time. There is a famous but apocryphal story told by a mathematician called Zeno called "Achilles and the Tortoise". You have probably heard of it so I'll be very brief.

Achilles gives the Tortoise a start in a race. Each stage of the infinite set of stages consists of 'Achilles gets to where the Tortoise was (at the previous stage) and the Tortoise is still in front'. For any Finite number of stages Achilles is always behind the Tortoise.

Infinity is different from a very large number. It has different properties. After any (as large as you like) large number of stages Achilles is behind the Tortoise. The situation is different if you consider an Infinite number of stages. After an Infinite number of stages Achilles is exactly level with the Tortoise.

We all know that Achilles does catch up with the Tortoise and does so within a Finite time. The conclusion has to be that the Infinite number of stages necessary for Achilles to draw level with the Tortoise can be executed within a Finite time.

Those of you following this discussion in detail will know that the Cardinality of the

number of stages in this example is not *C* but the 'old fashioned' ∞ type. Nevertheless, I can assure you that if we redesigned the story of Zeno so that the number of stages had the larger Cardinality *C*, this infinite set of stages could still be completed within a finite time.

Digital computers, by their very nature can not execute an infinite number of operations within a Finite time. The type of computer I and many others built in the 1960s could execute an infinite number of operations within a finite time. On such a computer the mathematical model of Achilles would catch the Tortoise – and overtake it! On such a computer an infinite number of stages having Cardinality *C* can be executed just as quickly and rather more easily.

For those of you who have been following my Column for some years I invite you to go back to the story of the bouncing ball. Briefly each bounce is smaller and takes a shorter time than the previous bounce. If the ball were replaced by a mathematical point which bounced would the ball ever stop bouncing? The answer is "Yes!" not because of any physical deficiency in the ball but because the ball (or at least a mathematical point) would be able to bounce a ∞ infinite number of times within a finite time.

An Instant in time has a Cardinality of either one or zero (I'm not quite sure which but I'd like to hear from you if you have any ideas about it) but of this I am sure, any interval of time larger than such an Instant has the same Cardinality as Eternity. Eternity can be mapped to a single second or less. So can the infinity of events which comprise the whole of Eternity!

The Cardinality of $\mathbb{C}$

I must start on my 'closing arguments' but, before I do, I have a question for those of you who know about the Cardinality of Infinite sets. Perhaps Peter can help me here. I shall put the question in three different ways. The Real numbers have Cardinality *C*. What is the Cardinality of $\mathbb{C}$? Is it *C* or is it more than *C*? Can a plane such as the Argand Diagram be mapped to the Number line? Is there a way in which a line such as the Number line can be made to fill an area such as the Argand Diagram?

You can reply to me by email or letter but please don't use the 'phone.

The 'Smoothie' Computer

From the correspondence I've received in the past I know that there will be many of you who will have used one of those 'smoothie' computers to which I referred earlier. It is equally likely that some of you will have no idea what I'm talking about. The 'smoothie' computer to which I have referred has a name. It is called an Analogue Computer. Perhaps the simplest Analogue Computer we all own is a HiFi set. The output of this simplest of all Analogue Computers is directly proportional to the input – but analogue computers can do much more than this!

Analogue Computers handle smoothly varying inputs with remarkable ease. Digital Computers can not and never will be able to do those things which depend on analysing the smoothness of functions. The best that they can do is to 'sample' the input at jerky intervals of time. The types of analysis which can be done using the mathematical

disciplines of Differential and Integral Calculus depend on the smoothness of the functions to which they can be applied. Analogue Computers handle Differentiation and Integration with complete and utter simplicity. They handle dynamic problems in real time or faster.

All those who have used Analogue Computers will know how easy it is to program them to analyse smoothly varying phenomena. Furthermore it is my guess that those of my readers who have programmed them will have mapped the infinite to a limited range of voltages at some time or other, maybe without even realising it.

Finally

Heineken has its uses for reaching those sets of numbers which other tools can not reach. Digital Computers have limitations which Analogue Computers do not have. Analogue computers are better at handling Infinite Sets and are particularly suited to mathematical analysis of smoothly varying functions.

Although I haven't mentioned it 'Functions of a Complex Variable' have a particular sort of smoothness which is called Regular. Regular functions have all sorts of interesting properties. Books about the nature of the physical universe from the theory of gravity to the more modern quantum electrodynamics are filled with Regular Complex Functions. My final question today is this: "Is the Digital Computer the wrong tool for creating mathematical models of the physical universe?"

My snail and email addresses are in Paul's Fact File at the end of Archive. Send me a stamp if you want a reply.