

Gerald's Column by Gerald Fitton

Over the last few months we have been looking at different sorts of number. As I have said mathematicians know of many types of number. However, up to now, I have concentrated on the first four types shown in the table below.

$$\begin{aligned}\mathbb{N} &= \{0, 1, 2, 3, 4, 5, 6, \dots\} &&= \text{Natural (counting) numbers} \\ \mathbb{Z} &= \mathbb{N} + \{0, -1, -2, -3, \dots\} &&= \text{Integers (whole numbers)} \\ \mathbb{Q} &= \mathbb{Z} + \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \dots\} &&= \text{Rational numbers (fractions)} \\ \mathbb{R} &= \mathbb{Q} + \{\pm\sqrt{2}, \pi, e, \dots\} &&= \text{Real numbers (smooth continuum)} \\ \mathbb{C} &= \mathbb{R} + \{\sqrt{-1}, (1 + \sqrt{-1}), \dots\} &&= \text{Complex (and imaginary) numbers}\end{aligned}$$

In the last couple of months I digressed to discuss 'groups of numbers'. I concentrated on Arrays and Matrices because nearly all modern spreadsheets support Array arithmetic and Matrix arithmetic, but I did mention two other types, quaternions and vectors.

It is time to move on to the set \mathbb{C} , the set known as Complex numbers. This month I shall discuss the numbers themselves leaving the subject 'Functions of a Complex Variable' (I don't mean spreadsheet functions) until next month.

Do it Backwards

One most fruitful mathematical technique is to try to invert an operation (do it backwards).

An example of inversion is the way in which we attempted to invert the addition of numbers from the set \mathbb{N} , the Natural numbers (the numbers we use for counting). We found that the inverse operation, subtraction, sometimes gave a result which was not a Natural number; it was a negative whole number. By inverting addition we discovered the set \mathbb{Z} , the Integers. The set of Integers, \mathbb{Z} , contains not only all the Natural numbers but negative whole numbers as well.

Then we looked at the inverse of multiplication, division. This inverse operation lead us to the discovery of fractions. Fractions are the new numbers which make the set of Rational numbers, \mathbb{Q} , different from the set \mathbb{Z} .

The set \mathbb{R} , the Real numbers, contains many more numbers than there are in the set \mathbb{Q} . The number of Rational numbers is infinite but this infinity is the sort which can be counted using the Natural numbers. However, the number of Real numbers is so large that the Natural numbers can not be used to count them! To use the jargon of mathematics, the number of Rational numbers is the \aleph sort of infinity whereas the number of Real numbers is of the \aleph sort! These two sorts of infinity are sometimes called 'denumerable' and 'non-denumerable' because the number of numbers in the set can either be counted or not counted using the Natural numbers.

I believe that it was a mathematician called Cantor who discovered that $\aleph + \aleph = \aleph$ and even that $\aleph * \aleph = \aleph$ but that when he tried \aleph^{\aleph} then he got a new sort of infinity which he called \aleph (pronounced aleph and sometimes written with a subscript). It is unfortunate that

some text books avoid using this character – I think it is Hebrew so perhaps they don't have it in their character sets. Instead they use the much less dramatic character \mathfrak{C} – and claim that it is short for the 'Cardinality of the Continuum'. It is the largeness of the \mathbb{R} set which gives it its smoothness, a topic to which I shall return next month.

Finding Square Roots

Finding the square of a real number is a straight forward multiplication sum. Finding the square root of a number is an inverse operation. If you square any number from the set \mathbb{R} the result will be a member of \mathbb{R} . Not so for the inverse operation, square root.

In an earlier article I commented that the symbol in front of the one in -1 is a unary operator and that -1 should be spoken of as 'negative one' rather than 'minus one'. The same symbol when used in the expression $(2 - 1)$ has an entirely different meaning – it is the 'minus' symbol. Let me remind you that the unary negative symbol has top priority whereas the minus symbol is below multiply in the order of precedence of operators.

The positive square root of negative one, $\sqrt{-1}$, can not be found amongst the Real numbers, \mathbb{R} , so, with this inverse operation, we've discovered another new number. In fact we've discovered a new set of numbers; this new set is given the symbol \mathbb{C} and is called the set of Complex numbers.

The positive square root of negative one is given the symbol i by mathematicians. It is given the symbol j by electrical engineers because they already use i as the symbol for electric current and don't want to get confused. I taught Complex numbers to both A level students (using i) and Electronic and Electrical Engineers (using j). I think I made the mistake of confusing them in my handouts only once – but that was most embarrassing!

Completeness

Another concept held dear by mathematicians is called Completeness.

Having discovered i , the positive square root of negative one, the question which needs to be answered is this. "If we take any Real number (from the set \mathbb{R}) and try to find its square root then can we find it either as a Real number or as a number which uses i , the positive square root of negative one?" If we can not then we need some more numbers besides i .

Let's take as an example the square root of negative four, $\sqrt{-4}$. Using the usual rules of arithmetic we can convert this. $\sqrt{4 * (-1)} = \sqrt{4} * \sqrt{-1} = 2 * \sqrt{-1} = 2 * i$. We have found the square root of negative four without having to use any new numbers. The answer to the question of the previous paragraph is that we can find the square root of any Real number and the answer will not need any new numbers other than the i to which we're already committed.

Now to the next question. "If we try to find the square root of any number with an i in it do we need a new number?". Let us take as an example \sqrt{i} . I'll give you the answer; the \sqrt{i} is $(1 + i)/\sqrt{2}$. You can prove this for yourself by squaring $(1 + i)/\sqrt{2}$ the long way. $(1 + i)^2 = 1 + 2*i - 1 = 2*i$. I know I haven't proved it but the answer to the question of this paragraph is that the square root of any number with an i in it needs no new numbers.

There is something even more important. I'm not going to prove to you mathematically (though it can be proved conclusively) but I will demonstrate by example with a spreadsheet a little later. It is this. You can take any root, say a fifth root (the inverse of the fifth power), you can find any function such as $\log()$ or $\tan()$ of any Complex number and the answer will not require any new number other than the i we know we need.

The Complex number set, \mathbb{C} , is complete. Whatever operation we carry out, whatever inverse operation we apply to a number belonging to \mathbb{C} the answer will always belong to \mathbb{C} . Let me put this another way, using the tool of inversion which has served us so well in our journey of discovery of new numbers, we have come to the end of the road. There are no new numbers to discover using any operation or its inverse that we know of now or which we might discover or invent in the future. Like our journey, \mathbb{C} is complete.

Notation

A complex number such as $(3 + 4*i)$ is sometimes written in longhand and in text books as an ordered pair of numbers, $(3, 4)$. Because I want to deal mainly with the way in which spreadsheets handle complex numbers I shall use the notation $\{3, 4\}$ for the complex number $(3 + 4*i)$ rather than $(3, 4)$. I shall do this because $\{3, 4\}$ is the notation used for storing the two parts of the complex number within a single cell.

The usual symbol for a Complex variable is z . The two components are usually given the symbols x and y . Hence, in my notation $z = x + i * y = \{x, y\}$. When I need a second Complex variable I shall use $w = \{u, v\}$.

Addition and Subtraction

The rule for the addition of Complex numbers is exactly the same as that for Array addition and Matrix addition. Incidentally, the rule for the addition of Vectors is also the same – so no surprises here.

The screenshot shows a spreadsheet window titled "PipeDream: SCSI::SQ_01.\$Archive.1998.9887.PipeDream.Complex01". The spreadsheet content is as follows:

| Complex Functions of Complex Numbers | | | | | |
|--|-----|----------------------|-----------|-------------------------|-----------|
| by Gerald L Fitton | | | | | |
| Input values into B10, C10, B11 and C11 only | | | | | |
| | | Using Real Functions | | Using Complex Functions | |
| | | Real | Imaginary | Real | Imaginary |
| 10 | z | 2.0000 | 1.0000 | 2.0000 | 1.0000 |
| 11 | w | 3.0000 | 0.0000 | 3.0000 | 0.0000 |
| 13 | z+w | 5.0000 | 1.0000 | 5.0000 | 1.0000 |

In the screenshot you will see a PipeDream spreadsheet in which I have added a couple of

Complex numbers. The rule for addition may be written as $z + w = \{x + u, y + v\}$.

The cell E13 contains the PipeDream function $c_add(z, w)$ where z and w are PipeDream Names for the contents of E10 and E11 respectively.

The Argand Diagram

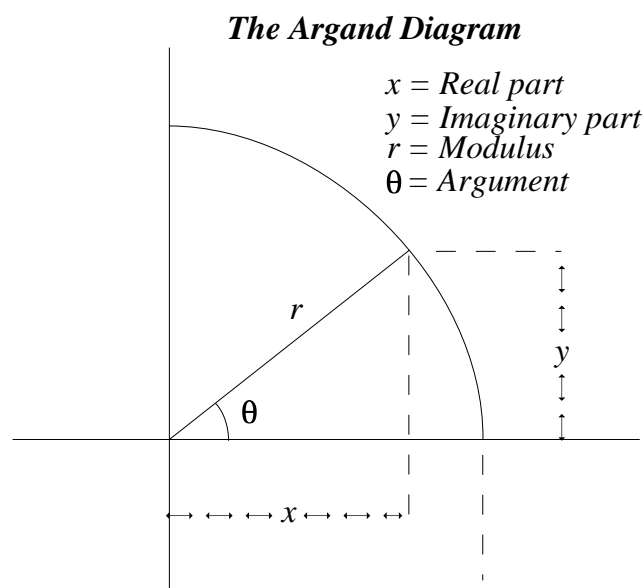
As you might have anticipated the rules for multiplication and division of Complex number arithmetic differ from those of Array, Matrix and Vector arithmetic. Before we investigate this we need to digress.

I shall describe a geometrical representation of Complex numbers but first let me quote from the article in which I introduced Real numbers:

“I have kept this Number Line until last because it is a concept which many can grasp. I hope that those of you who were becoming bewildered or even sceptical about the existence of many types of Real number will clutch at the Number Line. It can be used to check whether you have ascribed correct meanings to all the Un-Natural numbers which make up the set of Real numbers.”

Just as the Number Line came to our rescue as a geometrical representation of all Real numbers so the geometry of the two dimensional plane provides us with a means of visualising Complex numbers.

Contrary to any impression I might have given, Real numbers are not points on the Number Line. The true situation is better stated as “For every Real number there is a point on the Number Line – for every point on the Number Line there is a Real number.” Using the appropriate mathematical jargon we can say that there is one-to-one correspondence between the points on the Number Line and the Real numbers.



In a similar way there is a one-to-one correspondence between the points in a plane and the Complex numbers. This ‘Complex number plane’ is called after the man who originally discovered this relationship, Jean-Robert Argand (1768–1822), a Swiss mathematician born in Greece.

Jean-Robert Argand published his work in 1806 anonymously (in French); his identity was not revealed until six years later. One ‘obvious’ consequence of his discovery is that the points in the Argand Diagram can be denoted either by using either Cartesian or by using Polar co-ordinates. The Cartesian co-ordinates are x and y . The Polar co-ordinates are r and θ . The relationship is shown in the Argand Diagram. The Argument, θ is measured in radians and not in degrees.

Each format can be converted to the other. In the table below I am using the functions as they are named in PipeDream; some other spreadsheets use slightly different names for these functions.

$$x = r * \cos(\theta)$$

$$y = r * \sin(\theta)$$

$$r = \text{sqr}(x^2 + y^2) \text{ from Pythagoras' Theorem}$$

$$\theta = \text{atan}_2(y/x) \text{ note that } \text{atan}_2(y/x) \text{ can return values between } -\pi \text{ and } +\pi$$

Multiplication and Division

Using the Cartesian format the Rule for multiplication is:

$$z * w = \{(x * u) - (y * v), (x * v) + (y * u)\}$$

Of course this can be ‘proved’ by multiplying out the Complex numbers and substituting -1 for $(i * i)$ when it occurs.

I believe that a better way of visualising multiplication of z and w is to consider the numbers in Polar format. In words this is: “Multiply the modulus of z by the modulus of w ; add the argument of z to the argument of w ”. As a pair of formula it is:

$$\text{Modulus}(z * w) = (r_1 * r_2)$$

$$\text{Argument}(z * w) = (\theta_1 + \theta_2)$$

This representation can be converted back to the Cartesian form above but, in my view, the Polar representation of multiplication gives a better ‘feel’ for what is happening.

Using a Spreadsheet

Both PipeDream and Fireworkz support complex functions. The function in cell E15 of the screenshot shows the syntax of the function $c_mul(z,w)$ used to multiply z by w .

I suggest that you plot the two numbers z and w on the Argand diagram. They have the same Real part but conjugate Imaginary parts.

The Complex number z , $\{3, 4\}$ has a modulus $r = \text{sqr}(3^2 + 4^2) = 5$. The number w , $\{3, -4\}$ has the modulus $r = \text{sqr}(3^2 + (-4)^2) = 5$. The two modulus have the same value. The arguments are $\text{atan}_2(4/3)$ and $-\text{atan}_2(4/3)$. If you add these angles together you get zero so you’ll see that the product of z and w is a Real number.

The product of $\{3, 4\}$ and $\{3, -4\}$ is $\{5*5, 0\} = \{25,0\}$ as shown in the spreadsheet.

| | A | B | C | D | E | F |
|----|-----|--|-----------|---|-------------------------|-----------|
| 1 | | | | | | |
| 2 | | Complex Functions of Complex Numbers | | | | |
| 3 | | by Gerald L Fitton | | | | |
| 4 | | Input values into B10, C10, B11 and C11 only | | | | |
| 5 | | Using Real Functions | | | Using Complex Functions | |
| 6 | | | | | | |
| 7 | | | | | | |
| 8 | | Real | Imaginary | | Real | Imaginary |
| 9 | | | | | | |
| 10 | z | 3.0000 | 4.0000 | | 3.0000 | 4.0000 |
| 11 | w | 3.0000 | -4.0000 | | 3.0000 | -4.0000 |
| 12 | | | | | | |
| 13 | z+w | 6.0000 | 0.0000 | | 6.0000 | 0.0000 |
| 14 | | | | | | |
| 15 | z*w | 25.0000 | 0.0000 | | 25.0000 | 0.0000 |

Other Functions

Back in 1991 I needed to do some Complex number arithmetic for an Engineering course I was working on. By this I mean that the Engineers had to use a numerical method to solve an engineering problem. At that time PipeDream 4 did not exist – I had PipeDream 3 which does not support complex functions. So, to solve the problem with the PipeDream 3 spreadsheet I worked out how to find all the usual trigonometrical and hyperbolic functions of Complex numbers using Real functions. Of course, from my experience of designing Aeroplanes I knew that all these Complex functions could be written in terms of the exponential function. So that's what I did!

In columns B and C of the spreadsheet [Complex01] you will find fully worked examples of the identities which I used back in 1991; you will find the PipeDream 4 equivalent in column E.

De Moivre's Theorem

Abraham de Moivre (1667–1754) was a French mathematician who came to England at the age of 20 after the Revocation of the Edict of Nantes (he was a Protestant – look it up and guess at the consequences). The theorem named after him states that:

$$\exp(i * \theta) = \cos(\theta) + i * \sin(\theta)$$

There are many ways of proving this theorem – the 'best' proof depends on your starting point. I rather like the method which starts by finding the infinite series expansions of three functions (exp, sin and cos) using the Taylor or Maclaurin expansion. I have referred to these in an earlier article. The series for the exponential function is identically the sum of the series for cos and *i* times the series for sin.

It is this theorem which sets us on the road towards finding the numerical values of all the trigonometrical and hyperbolic functions of a complex variable.

Some Examples

You can study all the examples shown in the screenshot and many more by loading the PipeDream spreadsheet [Complex01] and changing the values of z and w . there are many more but out of the screenshot.

| | A | B | C | D | E | F |
|----|---------------------------|----------------------|-----------|---|-------------------------|-----------|
| 6 | | Using Real Functions | | | Using Complex Functions | |
| 7 | | Real | Imaginary | | Real | Imaginary |
| 8 | | | | | | |
| 9 | | | | | | |
| 10 | z | -1.0000 | 0.0000 | | -1.0000 | 0.0000 |
| 11 | w | 0.0000 | 1.0000 | | 0.0000 | 1.0000 |
| 12 | | | | | | |
| 13 | z+w | -1.0000 | 1.0000 | | -1.0000 | 1.0000 |
| 14 | z*w | 0.0000 | -1.0000 | | 0.0000 | -1.0000 |
| 15 | | | | | | |
| 16 | iz | 0.0000 | -1.0000 | | 0.0000 | -1.0000 |
| 17 | | | | | | |
| 18 | Complex powers | | | | | |
| 19 | ln z | 0.0000 | 3.1416 | | 0.0000 | 3.1416 |
| 20 | w*lnz | -3.1416 | 0.0000 | | -3.1416 | 0.0000 |
| 21 | z^w=exp(w*ln(z)) | 0.0432 | 0.0000 | | 0.0432 | 0.0000 |
| 22 | | | | | | |
| 23 | Trigonometrical functions | | | | | |
| 24 | exp(z) | 0.3679 | 0.0000 | | 0.3679 | 0.0000 |
| 25 | exp(-z) | 2.7183 | 0.0000 | | 2.7183 | 0.0000 |
| 26 | exp(iz) | 0.5403 | -0.8415 | | 0.5403 | -0.8415 |
| 27 | exp(-iz) | 0.5403 | 0.8415 | | 0.5403 | 0.8415 |
| 28 | | | | | | |
| 29 | sin(z) | -0.8415 | 0.0000 | | -0.8415 | 0.0000 |
| 30 | cos(z) | 0.5403 | 0.0000 | | 0.5403 | 0.0000 |

If your spreadsheet does not support complex functions in the way that PipeDream does then the two key formulae are those which give the Real and Imaginary parts of $\exp(z)$. For $w = \exp(z)$ these formulae are:

$$u = \exp(x) * \cos(y)$$

$$v = \exp(x) * \sin(y)$$

You will see the first of these in the screenshot. The value of $\exp(\{-1, 0\})$ is $\{0.3679, 0\}$. An interesting result is the value of $\ln(\{-1, 0\}) = \{0, \pi\}$. The inverse of this is that $\exp(\{0, \pi\}) = \{-1, 0\}$. Sometimes this is written as $\exp(i*\pi) = -1$, an identity considered to have almost mystical significance. Another interesting result is that the cosine of any Imaginary number is a Real number! As an example $\cos(\{0, -1\}) = \{1.5431, 0\}$. You will see that the cosine is a Real number greater than one, something which can not happen with angles that are Real numbers.

I suggest that you have a look in B19 and C19 which return the Real and Imaginary parts of the complex function $\ln(z)$. This is the starting point for raising numbers to complex powers—see the functions in row 21.

If you want to experiment further then I suggest that you do so with the file [Complex01] which you will find on the monthly disc and on the Abacus Training web site. If you want them directly from me then please send a formatted disc (800Kb preferred), a self addressed label and return postage.

Summary

The most important point I want to make is that the set of Complex numbers is mathematically Complete. By this I mean that you can not discover any new type of number by inverting any function of a Complex number. Whatever function of a Complex number you choose to investigate (or its inverse) the answer can always be expressed as a Complex number – no new type of number is required.

It was after Abraham de Moivre was released from a French prison (he was there because he was a Protestant) that he discovered the theorem named after him. If you want to use numerical methods to do Complex number sums (in these days you would use a spreadsheet rather than paper and pencil) then it is this theorem which is the starting point for developing all the formulae you will use in your spreadsheet. All these formulae are functions of Real variables and, as implied by the Completeness of the previous paragraph, the remarkable thing is that these Real functions always have an answer and the answer is always a Real number.

Because all the formulae are functions of Real variables you can do Complex number sums with any spreadsheet even if, unlike PipeDream and Fireworkz, they do not have Complex functions built-in. The Complex functions included in Pipedream and Fireworkz include all the trigonometrical and hyperbolic functions and the corresponding inverse functions. If you need to do Complex number sums then this facility will save you the trouble of working out these Real functions for yourself – and you are less likely to make a mistake.

It was Jean-Robert Argand who found a geometrical representation of Complex numbers. The Argand diagram, a two dimensional plane, is named after him. Multiplication and Division of Complex numbers is best visualised by considering the Polar form of the numbers in the Argand Diagram and by plotting the answers thereon.

Finally

‘Functions of a Complex Variable’ is a subject not to be confused with spreadsheet functions such as the $c_exp(z)$ of this article. I will say a few words about this subject next month and show you how a spreadsheet can be used for ‘mapping’ Complex functions to the Argand Diagram.

In the meantime, should you wish to contact me, you will find my details under the heading Abacus Training on the inside back cover of Archive.