Gerald's Column
by Gerald Fitton

The theme of my current series is: "How have mathematicians influenced the development of the computer?" Later, but not this month, I shall proceed to the question: "Is there another way in which the computer might be developed?"

It would seem that you are enjoying my description of the way in which different types of number were discovered (or, if you prefer, invented) and you are not fretting too much about my procrastination in not answering the second question.

Before I do anything else I must put on record my great pleasure to hear that Audrey will be renewing her writing for Archive. By Audrey I mean the one who is notorious for using one side of the bed – John being on the other. I have always found her articles are written with great style; much to Audrey's amusement I once used the adjective "winsome" to describe them – it was meant and taken as a compliment. I had thought of including a screen shot from my Psion machine as a token of my appreciation of her return but, after some consideration, I decided that a piece of prose might be more appropriate. A picture may paint a thousand words but there are occasions when a few well chosen words might have more impact; you will have to wait until the end of this article for the paragraphs I have dedicated to 'Audrey with the bedside Pocket Book'.

Mathematics, Invention or Discovery

The debate continues as to whether Mathematics was discovered or invented by Mankind. Those who favour invention usually maintain that negative integers do not exist "in the real world" whereas those who favour discovery (as I do) have given me some quite sophisticated examples of Mathematics being too strange a concept to have been invented. Typically one lady (who I guess is a pretty good mathematician) wrote "The fundamental nature of (higher) mathematics is so bizarre that no one could have invented it! It has to be part of reality!"

I shall save for another day the remarks made by you in response to my suggestion that "Our mathematical talent may be just a clouded image of the exceptional mathematical talent of our Creator."

The Future of Computers

I have been taken to task for saying that "computers do best when all relevant facts are available and when there is only one 'True' answer to a problem". I still believe that my remark is essentially correct and that the computer uses 'tricks' rather than a strategy when dealing with situations for which there are many answers having equal validity (or no answer at all).

My only comment at this stage is that children can learn by themselves how to do things which mathematically based computers would find excruciatingly difficult or even impossible. Riding a bicycle is only one such activity. "Except ye . . . become as little children" you will miss out on all sorts of things – the wondrous nature of negative numbers is but one; understanding how computers might develop is another.

Last Month's Numbers

You will remember the table shown below.

$\mathbb{N}$	=	$\{0, 1, 2, 3, 4, 5, 6, \ldots\}$		=	Natural (counting) numbers
$\mathbb{Z}$	=	$\mathbb{N}$	$+ \{0, -1, -2, -3, \ldots\}$	=	Integers (whole numbers)
$\mathbb{Q}$	=	$\mathbb{Z}$	$+ \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots\}$	=	Rational numbers (fractions)
$\mathbb{R}$	=	$\mathbb{Q}$	$+ \{\pm\sqrt{2}, \pi, e, \ldots\}$	=	Real numbers (smooth continuum)
$\mathbb{C}$	=	$\mathbb{R}$	$+ \{\sqrt{-1}, (1 + \sqrt{-1}), \ldots\}$	=	Complex (and imaginary) numbers

The Natural Numbers, $\mathbb{N}$, can be used for counting. Not only that, if all you want to do is to count then you don't need any other sort of number. More numbers, the negative integers which are included in the set $\mathbb{Z}$, arise by inverting the operation of Addition (doing it backwards) and insisting that the new numbers so discovered have a real existence.

Under the operations of Addition and its inverse Subtraction, the set of Integers, $\mathbb{Z}$, is complete in the mathematical sense. Here "complete" means that if you choose any pair of numbers from the set $\mathbb{Z}$ then the answer belongs to $\mathbb{Z}$. So have we exhausted all the possibilities? Do we have a need for any more numbers?

This Month

This month we shall try out a new operation, Multiplication and its inverse, Division. We shall apply these operations to the set $\mathbb{Z}$ and find out if it is sufficiently large to provide all possible answers. Of course we both know that $\mathbb{Z}$ is not complete under the operation of Division. We shall discover a new type of number, $\mathbb{Q}$, the Rational numbers (fractions).

Some of you are too familiar with the rules of arithmetic to appreciate the wondrous nature of negative numbers and fractions so, along the way, I shall try to convince you that those prominent philosophers who rejected them as 'Un-Natural' do have a point worth considering.

Multiplication is the name of the game

If you start with any pair of integers, positive or negative, and multiply them together then without exception your answer will be an integer from the set $\mathbb{Z}$. There are no new numbers to be found by introducing the operation of Multiplication.

Multiplication is essentially multiple 'piles of' the same number of items. By this I mean that if you start with, say, five piles each having three counters you might want to know how many counters you have all together. Five piles each having exactly three counters makes a total of fifteen counters. You can find out the answer by evaluating 3+3+3+3+3, adding together (merging) five piles of three into one big pile of fifteen. The mathematical expression of this is $3 \times 5 = 15$. Please note that the order used in mathematical notation is 'number in a pile' the times symbol ($\times$), and finally the 'number of piles'.

I have an anecdote to relate to you here. When I was about five years old I discovered that five piles of three contained exactly the same number of counters as did three piles of five. I got all excited about this and tried again with other pairs of natural numbers. It always worked out. I tried to explain to my teacher what I'd just discovered. To me the concept that 'piles' and 'number in a pile' were interchangeable was something mystical (or at least wondrous) but I think that to the teacher it was so obvious that she couldn't see why I was so worked up about it. I even went on to develop what I would now call a proof that multiplication is commutative (reversible) for any pair of natural numbers – but the teacher remained unimpressed.

What you may find more difficult to visualise is the meaning of $-3 \times 5 = -15$ in terms of piles having unary negative three units in each pile. Let me have a go. To each of five people you owe three pounds. How much credit do you have? Your total credit can be calculated by continued addition as before, $(-3)+(-3)+(-3)+(-3)+(-3)$. The positive symbols are all operators (add together) whereas the negative signs are all unary minus. Now you have a go at $(+5)\times(-3)$; negative three piles each containing +5 units!

If you can conceive of negative three piles each containing negative five units $(-5)\times(-3)$ and discovering that you have a total of +15 units then you are well on the road to believing that negative numbers exist in the real world. If all you can do is to use a rote learned rule such as "minus times a minus is a plus" then you aren't quite there yet! You need to "become as little children" and question the very existence of negative numbers before you can begin to appreciate how wonderful they are.

In the computer

I have told you that inside a computer negative numbers are represented by large positive numbers. Instead of doing the usual negative number arithmetic, the sum is done in modular or 'clock' arithmetic; it always works. The computer in its present form can not store negative numbers so instead, it adds together the large numbers that represent negative numbers using modular arithmetic.

I have been asked to give an example of the 'clock' arithmetic taught to my six year old grandson, Craig, at school. Here we go!

It is six o'clock in the morning and you went to bed eleven hours ago. What time did you go to bed? Now we all know that 6 minus (verb) 11 is negative (adjective) 5 but there is no such time as 'Negative five o'clock'. The hours shown by a clock are all natural numbers and the arithmetic which yields '7 pm the evening before' as the answer is a form of arithmetic which uses as its modulus the number twelve.

As another example I shall use decimal notation modulus 1000 to add together three lots of negative five. In modulus 1000 decimal notation the number negative five is 995. Three lots can be added as 995+995+995. The answer using the normal rules of arithmetic is 2985. The next step is to apply the modulus 1000 operator which means that we throw away the 2 (in the thousands column) to leave the answer as 985. In the notation we are using 985 corresponds to negative 15.

Now to multiplication (not of but) by negative numbers. The notation $5\times(-3)$ means start with zero (or 1000 if you prefer) and subtract three lots of positive five. That is not too difficult but, as they say in the best text books I shall leave as an exercise for the reader the execution of negative three times negative five, written as $(-5)\times(-3)$ in the usual notation, but written as 995×997 in modulus 1000 decimal notation. Hint: Do the sum in the usual way and then get rid of the unwanted leading digits in the thousands column and above.

Let me summarise the point I'm trying to make. Computers do not store negative numbers; instead they use large positive numbers to represent negative numbers. Computers do not multiply by negative numbers; instead they multiply by large positive numbers using modular arithmetic. Using modular arithmetic, you do the sum in the usual way and then throw away digits which appear to the left of your chosen most significant position, you might say that these extra digits overflow out of the calculation. When you have done that you can tell whether your answer is a negative or positive number by its size. Large numbers correspond to negative numbers!

I don't think my six year old Craig uses negative numbers at all when doing 'clock' arithmetic. From my questions to him I think that when he subtracts eleven hours from 6 am to discover his bed time he adds the 12 hours at a much earlier stage. Consequently he is subtracting 11 hours from 18 hours and not adding 12 hours to negative 5 hours.

Recapitulation

If the only operations you need to do with numbers on a computer were Addition, Subtraction and Multiplication of integers from the set $\mathbb{Z}$ then the only numbers you would need to store in the computer would be natural numbers from the set $\mathbb{N}$. This set of natural numbers would be divided into two equally sized subsets which I might call 'small' and 'large'. The 'large' subset would represent the negative numbers.

Early personal computers such as the Dragon and Clive Sinclaire's Z80 (or was it a Z81?) had a Basic language which would do sums only with integers but they would handle negative integers. The early BBC, the early Apple and IBM machines would do more.

Doing it backwards

I have referred before to 'doing it backwards'. In mathematical jargon 'doing it backwards' is called inverting the operation! The inverse of Multiplication is Division. Multiplication might be considered to be 'piles of' and Division 'sharing out'.

Five piles of three counters makes fifteen counters. Suppose we start with fifteen counters and we want to share it out into five equal piles. This operation is called Division. I'm sure that you can conceive of many ways of doing this with fifteen counters and five piles. For example you can deal out the counters as you might a deck of fifteen cards one round at a time to the five players. The question you have to answer is "How many cards does each player finish up with when the whole deck of fifteen has been dealt out?" The answer is three. Easy! No problem!

A more difficult concept is dealing out negative fifteen counters amongst five players. One way of looking at it is that five bankers may agree to take equal losses when a country goes bust owing fifteen billion somethings! Each banker becomes liable for three billion.

Even harder is sharing out negative fifteen 'widgets' amongst negative five participants. The participants (who exist only as negative entities) finish up with positive three 'widgets' each!

Now, which camp are you in? Are you a believer in the reality of negative numbers or are you experiencing the beginnings of scepticism? I hope that I am arousing scepticism in those of you whose arithmetic with negative numbers is flawless but is carried out by using a set of half understood rules such as "minus times a minus is a plus". Once you have got past simply using the rules rather than understanding them you are on the way towards that wonderment to which I have already referred.

A New Type of Number

Now we come to a most interesting question. It is this. If you take any pair of integers with all the positive and negative integers to choose from, and you divide one by the other is the answer always an integer from the set ℤ? Of course the answer is "No!" If you divide, say one by twelve then the answer is not an integer, it is the fraction we call one twelfth. If you have only one card and twelve players in a game then you will be hard put to share out the card amongst the twelve players. However, if you divide one hour by twelve you get one twelfth of an hour. The number one twelfth is not an integer but it does have a meaning and a value. The Babylonians divided the hour into sixty equal parts and not ten because sixty is divisible by many more numbers than is ten. There are more cases when you can share out the hour equally and finish up with an integer number of minutes than if the hour were divided into ten equal parts.

We have decimalised our currency and we are in the process of decimalising our weights and distance measures but I reckon it will be a long time before anyone will dare to mess around with the units of time!

Numbers such as one twelfth are called Rational Numbers and the set of all such numbers is given the symbol ℚ. I always thought ℚ stood for 'quotient' but I've been told that isn't so. I don't intend to prattle on about whether such numbers exist in the same way that I have about negative numbers because, by now, those of you who are interested will be able to develop your own thought experiments and those who are not interested might get bored.

Here is a valid statement. If you start off with any pair of rational numbers from the set ℚ, positive or negative, and execute the operations of Addition, Subtraction, Multiplication and Division then the answer is always a rational number from the set ℚ. The set ℚ is complete under the usual operations of arithmetic. So have we discovered all the numbers that there are? We shall see.

Storing rational numbers

I have said before that computers can store many integers from the set ℤ but, to do so, they have to resort to the trick of representing negative numbers by large natural numbers from the set ℕ. What about storing Rational numbers? Computers can store Rational numbers only by using a trick which converts the Rational numbers to a pair of Natural numbers from the set ℕ. I shall postpone until next month the details of how this is done.

I must emphasise one point you might have missed when you read that last sentence. You need two Natural numbers and not just one to represent a range of Rational numbers.

You can not store all Rational numbers in a computer. Those of you who have not read any of my previous articles may be surprised to know that a computer can store a quarter, an eighth and a sixteenth with perfect accuracy but can not store one tenth exactly. (But is the Pocket Book different?) I will deal with this another day because I have one more point to make this month and space is short.

More numbers?

Well, we can add, subtract, multiply and divide any pair of Rational numbers and the answer is always a Rational number. Are there any more types of number? Yes! There are!

There is quite a simple proof that the solution to the equation ***x^2 = 2*** can not be satisfied by any rational number. So, with the numbers we have up to now we can't invert the 'squaring' operation. We can't find square roots – we can't yet do that operation backwards.

A Rational number can always be expressed as a fraction of two Integers from the set $\mathbb{Z}$. If the square root of 2 were rational then we could start with the equation ***(n/m)^2 = 2*** and try to find the Integers n and m. The first step in the proof is to show that n must be even; it follows later that m must also be even. The proof continues by dividing both top and bottom of the fraction n/m by 2 (to get a couple more integers) and continuing for ever. I won't go into details because the proof was part of my 'O Level' syllabus (the 'O Level' was the pre cursor of the GCSE exams) and so I know that you'll be able to find it in any text book which is old enough.

Also it can be shown that the number written as π (and pronounced 'pie' – its value is the ratio of the circumference of a circle to its diameter) can not be Rational. Numbers such as the square root of 2 are called Irrational and those such as π are called Transcendental. The word 'Irrational' was chosen to describe numbers like the square root of two because it was considered to be irrational (mad) to believe in the existence of such numbers.

Finally it can be shown that there are many more Transcendental numbers than all the rest put together so there are quite a lot of numbers outside the set $\mathbb{Q}$! In the jargon you can 'count' how many Rational numbers there are by using the Integers but the set of Integers is just not big enough to count all the Transcendental numbers there are.

The set of all Rational, Irrational and Transcendental numbers is called the set of Real numbers and given the symbol $\mathbb{R}$.

The Number Line

It might be difficult to visualise negative numbers and fractions and even harder to recognise that there is a massive infinity of Transcendental numbers between every adjacent pair of fractions (Rational numbers).

Don't despair. Geometry comes to our rescue. Draw a straight line which stretches out for ever in both directions and, somewhere close to your own location, you'll find a zero marked on the line. Near to the zero are markings such as +1, +2, −1 and −2 at equal intervals along the line. This concept is called the Number Line. You can use it to do sums such as $3 - 5$ and get the right answer, −2.

Between the Integer markings are other points, some have labels such as one twelfth but others have labels such as the square root of 2 and π. It is not too difficult to draw a line exactly root 2 units long. Construct a 45 degree right angled triangle with the sides containing the right angle being 1 unit long. The hypotenuse is exactly root 2 units long and can be marked off on the Number Line.

Every point on the Number Line represents a unique Real number from the set $\mathbb{R}$. Every Real number from the set $\mathbb{R}$ has a corresponding point on the Number Line. There are no points left over; there are no numbers left over. There is what is called a one-to-one correspondence between the Real numbers, $\mathbb{R}$, and the Number Line. In mathematical jargon the Number Line is 'Continuous'; there are no gaps and no Real numbers without a place on the line.

I have kept this Number Line until last because it is a concept which many can grasp. I hope that those of you who were becoming bewildered or even sceptical about the existence of many types of Real number will clutch at the Number Line. It can be used to check whether you have ascribed correct meanings to all the Un-Natural numbers which make up the set of Real numbers.

In the computer

The 'Continuum' became all important to mathematicians and its understanding allowed the development of the differential and integral calculus. I shall say more on this subject another day. Digital computers can not store any representation of the continuum. Analogue computers can and do store, if only for an instant, any Real number including transcendental numbers such as π.

So what numbers can digital computers store? Certainly they can not store all the Real numbers, indeed they can not store any Irrational or Transcendental numbers accurately but only approximations to them. Computers can not store representations of all Rational numbers; for example it can not store one tenth accurately. Even when the computer comes across a Rational number it can store it has to use a pair of Natural numbers to represent such a Rational number.

By the way, in the BBC Basic handbook the distinction is made between Integers and Real numbers. I have to tell you that those which are called 'Real' are not Real but only a subset of the Rational numbers; we shall determine the extent of this subset next month.

Even more numbers

Have a look back at the diagram from the beginning of this article. You will see that after discovering all the different types of numbers we have right up to the Real numbers, $\mathbb{R}$, there are still some left, numbers from the set $\mathbb{C}$. These are the most exotic of all numbers.

They can not be mapped to the Number Line. There are many prominent mathematicians who will tell you that the numbers from the set ℂ are the numbers used by the Creator to construct the physical universe and that there is no way of understanding the nature of the physical universe without understanding these numbers first. In a future article I shall help you to discover these exotic numbers used by the Creator and introduce you to a spreadsheet (or two) which will handle approximations to numbers from the set ℂ.

I can not resist adding that analogue computers could not operate if they did not have built into them numbers from the set ℂ.

The Pocket Book Rules – “OK”!

The following is a small offering intended to celebrate the return of Audrey (who writes so winsomely from one side of the bed).

It was 10.00 am. Jill (from the other side of my bed) was asleep in the bridal suite of a hotel in Niagra Falls. How we got placed in the luxurious but expensive bridal suite is another story. She was still recovering from catching an aeroplane from New York at 4.00 am the previous day. I was suffering withdrawal symptoms. I hasten to add that my most urgent desire was not a cup of tea nor was it to awaken Jill but to type something into my Archimedes which was sitting 3000 miles away. Then I found a ‘fix’. I had brought along my Pocket Book II. It quite accurately told me the time of sunrise and sunset in Niagra Falls, the time in the UK (about 3 pm) and, without fault, converted all our Canadian and US expenditure into sterling. By the way, if you ever go to Niagra Falls, Victoria Park on the Canadian side is magical.

Not wishing to wake Jill, despairing from my withdrawal symptoms and with the idleness which I enjoy whilst on holiday, I typed into the Abacus spreadsheet of my Pocket Book II the following formula: ***=if(0.1 + 0.1 + 0.1 − 0.3 = 0,"OK","Error")*** [Note that the smart quotes have been turned OFF]. The answer it returned was “OK”.

Jill was fast asleep; nevertheless the impact of this answer caused my brain to change up a gear and in sympathy she awakened. So what, you might wonder, is so remarkable about the answer “OK”? Isn’t it what I would have expected?

You try it in your favourite spreadsheet and see what you get! If you don’t have a spreadsheet then try it using the Basic64 program in this directory. Just double click on the [!RunBasic] file

So my challenge to followers of Audrey’s somewhat winsome column is “Why is the Pocket Book II superior to the Archimedes?”